[REVISION TOOL · JEE MAIN + ADVANCED · CLASS 11]

CONCEPT MAP

SEQUENCE AND SERIES

4 CHAPTERS · 17 CORE IDEAS · ONE SHEET

SCAN → PRACTISE

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REAL STUDY, GENUINE RESULTS.

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A ARITHMETIC PROGRESSION 01 -- 06

01 ARITHMETIC PROGRESSION (AP)

A · 01

A. DEFINITION

A sequence $a_1, a_2, a_3, \dots, a_n, \dots$ is called an arithmetic progression if the difference between any term and the term just before it is always the same, that is

$$a_n - a_{n-1} = \text{constant} = d \ \ (\text{independent of } n)$$

The fixed number d is called the common difference.

B. GENERAL TERM

For an AP with first term a and common difference d,

KEY

$$a_n = a + (n-1)d$$

If the AP has a finite number of terms, say m, then the nth term from the end is the $(m-n+1)$th term from the beginning:

$$a_{m-n+1} = a + (m-n)d$$

02 PROPERTIES OF TERMS OF AN AP

A · 02

i. A sequence is an AP if and only if its nth term is linear in n, i.e. $a_n = An + B$. In that case the coefficient of n is the common difference, so $A = d$.

ii. In a finite AP $a_1, a_2, \dots, a_n$ the sum of terms equidistant from the two ends is always the same and equals the sum of the first and last term:
$a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \dots$
that is, $a_k + a_{n-k+1} = a_1 + a_n$ for all $k = 1, 2, \dots, n$.

iii. If terms of an AP are picked at regular intervals, they again form an AP.

iv. Three numbers a, b, c are in AP if and only if $2b = a + c$.

v. If a_n, a_{n+1}, a_{n+2} are three consecutive terms of an AP, then $2a_{n+1} = a_n + a_{n+2}$.

vi. Twice any term of an AP equals the sum of any two terms equidistant from it: $2a_n = a_{n-k} + a_{n+k}, \ \forall\, k = 1, 2, 3, \dots$

vii. If a constant is added to, or subtracted from, every term of an AP, the new sequence is again an AP with the same common difference.

viii. If every term of an AP is multiplied or divided by a non-zero constant k, the new sequence is an AP with common difference kd or d/k.

03 SELECTION OF TERMS OF AN AP

A · 03

While solving problems it helps to choose the terms symmetrically. The usual choices are:

NUMBER OF TERMS	TERMS	COMMON DIFFERENCE
3	$a-d,\ a,\ a+d$	d
4	$a-3d,\ a-d,\ a+d,\ a+3d$	$2d$
5	$a-2d,\ a-d,\ a,\ a+d,\ a+2d$	d
6	$a-5d,\ a-3d,\ a-d,\ a+d,\ a+3d,\ a+5d$	$2d$

04 SUM OF n TERMS OF AN AP

A · 04

The sum S_n of the first n terms of an AP with first term a and common difference d is

KEY

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Equivalently, in terms of the first and last term,

$$S_n = \frac{n}{2}[a_1 + a_n], \quad a_1 = a$$

05 USEFUL RESULTS ON THE SUM OF AN AP

A · 05

i. A sequence is an AP if and only if the sum of its first n terms has the form $An^2 + Bn$, where A and B are constants. In that case the common difference equals $2A$.

ii. If the ratio of the sums of n terms of two APs is given, the ratio of their nth terms is obtained by replacing n by $2n-1$ in the given ratio.

iii. If the ratio of the nth terms of two APs is given, the ratio of the sums of their n terms is obtained by replacing n by $\frac{n+1}{2}$ in the given ratio.

06 INSERTION OF n ARITHMETIC MEANS

A · 06

Suppose n quantities $A_1, A_2, \dots, A_n$ are inserted between a and b so that

$$a,\ A_1,\ A_2,\ \dots,\ A_n,\ b \ \text{ is an AP}$$

These are the n arithmetic means between a and b. Let d be the common difference of this AP of $n+2$ terms. Then b is the $(n+2)$th term, so

KEY

$$b = a + (n+1)d \ \Rightarrow\ d = \frac{b-a}{n+1}$$

Hence

$$A_1 = a + d,\ A_2 = a + 2d,\ \dots,\ A_n = a + nd$$

Adding all of them,

$$A_1 + A_2 + \dots + A_n = \frac{n}{2}(A_1 + A_n) = \frac{n}{2}(a+b)$$

so the sum equals n times the single arithmetic mean between a and b.

Remark. If a single arithmetic mean A is inserted between a and b, then $A = \frac{a+b}{2}$, which is exactly the mid-point of the segment joining $(a, 0)$ and $(b, 0)$. If two arithmetic means A_1, A_2 are inserted, they divide that segment in the ratios $1:2$ and $2:1$, giving $A_1 = \frac{2a+b}{3}$ and $A_2 = \frac{a+2b}{3}$.

B GEOMETRIC PROGRESSION 07 -- 11

07 GEOMETRIC PROGRESSION (GP)

B · 07

A. DEFINITION

A sequence of non-zero numbers is called a geometric progression if the ratio of every term to the term just before it is always the same constant.

B. GENERAL TERM

If a is the first term and r the common ratio, the GP is $a,\ ar,\ ar^2,\ ar^3, \dots$ and

KEY

$$a_n = ar^{n-1}$$

For a finite GP of m terms, the nth term from the end is the $(m-n+1)$th term from the beginning, so it equals ar^{m-n}.

Remark. If the last term of a GP is l and the common ratio is r, the nth term from the end is $l\left(\frac{1}{r}\right)^{n-1}$.

08 SELECTION OF TERMS OF A GP

B · 08

Three or more terms of a GP may be chosen as follows:

NUMBER OF TERMS	TERMS	COMMON RATIO
3	$\frac{a}{r},\ a,\ ar$	r
4	$\frac{a}{r^3},\ \frac{a}{r},\ ar,\ ar^3$	r^2
5	$\frac{a}{r^2},\ \frac{a}{r},\ a,\ ar,\ ar^2$	r

09 PROPERTIES OF TERMS OF A GP

B · 09

i. If every term of a GP is multiplied or divided by the same non-zero constant, the result is again a GP with the same common ratio.

ii. The reciprocals of the terms of a GP form a GP.

iii. If each term of a GP is raised to the same power, the resulting sequence is again a GP.

iv. In a finite GP the product of terms equidistant from the two ends is always the same and equals the product of the first and last term.

v. Three non-zero numbers a, b, c are in GP if and only if $b^2 = ac$.

vi. If a, b, c are in GP, then b is the geometric mean of a and c.

vii. If $a_1, a_2, \dots, a_n$ are n non-negative numbers, their geometric mean is $G = (a_1 a_2 \cdots a_n)^{1/n}$.

viii. If terms of a GP are chosen at regular intervals, the new sequence is again a GP.

ix. If $a_1, a_2, a_3, \dots$ is a GP of positive terms, then $\log a_1, \log a_2, \log a_3, \dots$ is an AP, and conversely.

10 SUM OF n TERMS OF A GP

B · 10

For a GP with first term a and common ratio r,

KEY

$$S_n = \begin{cases} a\left(\dfrac{r^n - 1}{r-1}\right), & r \neq 1 \\ na, & r = 1 \end{cases}$$

In terms of the last term $l = ar^{n-1}$,

$$S_n = \frac{a - lr}{1-r} = \frac{lr - a}{r-1}, \qquad r \neq 1$$

The sum of an infinite GP with first term a and common ratio r, where $-1 < r < 1$, is

$$S_\infty = \frac{a}{1-r}$$

11 INSERTION OF n GEOMETRIC MEANS

B · 11

Let $G_1, G_2, \dots, G_n$ be inserted between two positive numbers a and b so that

$$a,\ G_1,\ G_2,\ \dots,\ G_n,\ b \ \text{ is a GP}$$

This GP has $n+2$ terms, so b is its $(n+2)$th term:

KEY

$$b = ar^{n+1} \ \Rightarrow\ r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

Therefore

$$G_1 = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, \quad G_2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, \ \dots, \quad G_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Note. (i) If a single geometric mean G is inserted between a and b, then $G^2 = ab$, so $G = \sqrt{ab}$.
(ii) The product of the n geometric means inserted between a and b equals the nth power of the single geometric mean:
$G_1 \cdot G_2 \cdot G_3 \cdots G_n = \left(\sqrt{ab}\right)^n$

C HARMONIC PROGRESSION & MEANS 12 -- 15

12 IMPORTANT PROPERTIES OF AM AND GM

C · 12

i. If A and G are respectively the arithmetic and geometric means between two positive, unequal numbers a and b, then $A > G$.

ii. If A and G are the arithmetic and geometric means between two positive numbers a and b, then a and b are the roots of the quadratic equation $x^2 - 2Ax + G^2 = 0$

iii. If A and G are the AM and GM between two positive numbers, then those numbers are $A \pm \sqrt{A^2 - G^2}$

iv. If the AM and the GM between two numbers are in the ratio $m : n$, then the numbers themselves are in the ratio $\left(m + \sqrt{m^2 - n^2}\right) : \left(m - \sqrt{m^2 - n^2}\right)$

13 HARMONIC PROGRESSION (HP)

C · 13

A sequence $a_1, a_2, a_3, \dots, a_n, \dots$ of non-zero real numbers is called a harmonic progression if

KEY

$$\frac{1}{a_1},\ \frac{1}{a_2},\ \frac{1}{a_3},\ \dots,\ \frac{1}{a_n},\ \dots \ \text{ is an AP}$$

For example $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \dots$ is an HP, because $1, 3, 5, 7, 9, \dots$ is an AP.

The nth term of an HP is the reciprocal of the nth term of the corresponding AP. So if $a_1, a_2, a_3, \dots$ is an HP and d is the common difference of the corresponding AP, then

$$d = \frac{1}{a_2} - \frac{1}{a_1}, \qquad a_n = \frac{1}{a + (n-1)d}, \quad a = \frac{1}{a_1}$$

14 INSERTION OF n HARMONIC MEANS

C · 14

Let a and b be two given numbers and let $H_1, H_2, \dots, H_n$ be inserted so that $a, H_1, H_2, \dots, H_n, b$ is an HP. Then

$$\frac{1}{a},\ \frac{1}{H_1},\ \frac{1}{H_2},\ \dots,\ \frac{1}{H_n},\ \frac{1}{b} \ \text{ is an AP}$$

Let D be the common difference of this AP. Since $\frac{1}{b}$ is its $(n+2)$th term,

KEY

$$\frac{1}{b} = \frac{1}{a} + (n+1)D \ \Rightarrow\ D = \frac{a-b}{(n+1)ab}$$

Hence

$$\frac{1}{H_1} = \frac{1}{a} + D, \quad \frac{1}{H_2} = \frac{1}{a} + 2D, \ \dots, \quad \frac{1}{H_n} = \frac{1}{a} + nD$$

Substituting the value of D gives $H_1, H_2, \dots, H_n$.

15 PROPERTIES OF AM, GM AND HM

C · 15

For two positive numbers a and b, the arithmetic mean A, geometric mean G and harmonic mean H are

KEY

$$A = \frac{a+b}{2}, \qquad G = \sqrt{ab}, \qquad H = \frac{2ab}{a+b}$$

These means satisfy:

i. $A \geq G \geq H$

ii. $G^2 = AH$, that is, A, G, H are in GP.

If A, G, H are the arithmetic, geometric and harmonic means between three numbers a, b, c, then a, b, c are the roots of

$$x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$$

D SPECIAL SEQUENCES & SERIES 16 -- 17

16 ARITHMETICO-GEOMETRIC SEQUENCE

D · 16

If $a_1, a_2, a_3, \dots$ is an AP and $b_1, b_2, b_3, \dots$ is a GP, then the sequence

$$a_1 b_1,\ a_2 b_2,\ a_3 b_3,\ \dots$$

is called an arithmetico-geometric sequence (AGP).

For example $a,\ (a+d)r,\ (a+2d)r^2,\ (a+3d)r^3, \dots$ is an arithmetico-geometric sequence.

Its general term is the product of the nth terms of the corresponding AP and GP:

$$T_n = [a + (n-1)d]\, r^{n-1}$$

The sum of n terms of $a,\ (a+d)r,\ (a+2d)r^2, \dots$ is

KEY

$$S_n = \begin{cases} \dfrac{a}{1-r} + dr\,\dfrac{1 - r^{n-1}}{(1-r)^2} - \dfrac{[a + (n-1)d]r^n}{1-r}, & r \neq 1 \\ \dfrac{n}{2}[2a + (n-1)d], & r = 1 \end{cases}$$

If $|r| < 1$, then $r^n \to 0$ and $r^{n-1} \to 0$ as $n \to \infty$, so the sum of the infinite arithmetico-geometric sequence is

$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

17 SUM OF n TERMS OF SOME SPECIAL SERIES

D · 17

i. Sum of the first n natural numbers:
$\sum_{k=1}^{n} k = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$

ii. Sum of the squares of the first n natural numbers:
$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

iii. Sum of the cubes of the first n natural numbers:
$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \dots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 = \left(\sum_{k=1}^{n} k\right)^2$

iv. Sum of the fourth powers of the first n natural numbers:
$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

METHOD OF DIFFERENCES

i. If the differences $T_2 - T_1,\ T_3 - T_2,\ T_4 - T_3, \dots$ between consecutive terms of a series $T_1 + T_2 + \dots + T_n$ form an AP, then the nth term is $T_n = an^2 + bn + c$, where a, b, c are constants.

ii. If those differences are in GP with common ratio r, then $T_n = ar^{n-1} + bn + c$

iii. If the differences of the differences of $T_2 - T_1,\ T_3 - T_2,\ T_4 - T_3, \dots$ are
(a) in AP, then $T_n = an^3 + bn^2 + cn + d$
(b) in GP, then $T_n = ar^{n-1} + bn^2 + cn + d$