CONCEPT MAP

QUADRATIC EQUATIONS

4 TOPICS · 19 CORE IDEAS · ONE SHEET

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SCAN TO PRACTICE

A EQUATION, ROOTS & DISCRIMINANT 01-05

01 QUADRATIC EQUATION AND ITS ROOTS

A·01

A. DEFINITION

An equation of the form

KEY

$$ax^2+bx+c=0, \qquad a\neq 0$$

where a, b, c are real (or complex) numbers, is called a quadratic equation in x. The number a is the leading coefficient and c is the constant term.

B. ROOT OF AN EQUATION

A number α is a root of $f(x)=0$ if $f(\alpha)=0$, that is $a\alpha^2+b\alpha+c=0$. A root is also called a zero of the expression ax^2+bx+c.

A quadratic equation has exactly two roots, counted with multiplicity, and they always exist in the set of complex numbers.

C. DEGENERATE CASES

- i. If $a=0$ and $b\neq 0$ the equation is linear and has the single root $x=-c/b$.
- ii. If $a=b=0$ and $c\neq 0$ there is no root.
- iii. If $a=b=c=0$ every value of x satisfies it, so it is an identity.

> **Important.** If a quadratic equation is satisfied by more than two distinct values of x, then it cannot be a genuine quadratic. In that case $a=b=c=0$ and the expression vanishes identically. This is the standard way to prove that an expression is an identity.

02 METHODS OF SOLVING

A·02

A. FACTORISATION

Split the middle term so that $ax^2+bx+c=a(x-\alpha)(x-\beta)$, then equate each factor to zero.

B. COMPLETING THE SQUARE

$$a\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a}$$

C. QUADRATIC FORMULA

Solving the equation above for x gives the general formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Writing $D=b^2-4ac$ for the discriminant, the two roots are

KEY

$$\alpha=\frac{-b+\sqrt{D}}{2a}, \qquad \beta=\frac{-b-\sqrt{D}}{2a}$$

03 NATURE OF THE ROOTS

A·03

Everything depends on the discriminant

KEY

$$D=b^2-4ac$$

Let a, b, c be real numbers with $a\neq 0$. Then:

- i. $D>0$: the roots are real and distinct.
- ii. $D=0$: the roots are real and equal, each being $-\frac{b}{2a}$.
- iii. $D<0$: the roots are non-real and form a conjugate pair $p\pm iq$.
- iv. $D\geq 0$: the roots are real. This single condition covers both of the first two cases.

RATIONAL COEFFICIENTS

Let a, b, c be rational. Then:

- i. $D>0$ and D a perfect square: the roots are rational and distinct.
- ii. $D>0$ and D not a perfect square: the roots are irrational and occur as a conjugate pair $p+\sqrt{q}$ and $p-\sqrt{q}$.
- iii. $a=1$, b and c integers and D a perfect square: the roots are integers.

> **Caution.** The conjugate-pair results need real, or rational, coefficients. If a, b, c are allowed to be complex, the roots need not be conjugates and $D<0$ no longer means anything.

04 RELATION BETWEEN ROOTS AND COEFFICIENTS

A·04

If α and β are the roots of $ax^2+bx+c=0$, then $ax^2+bx+c=a(x-\alpha)(x-\beta)$, and comparing coefficients gives

KEY

$$\alpha+\beta=-\frac{b}{a}, \qquad \alpha\beta=\frac{c}{a}$$

The difference of the roots follows from the formula:

$$\alpha-\beta=\pm\frac{\sqrt{D}}{a}, \qquad (\alpha-\beta)^2=\frac{D}{a^2}$$

> **Use.** These two relations turn almost every root problem into algebra in $S=\alpha+\beta$ and $P=\alpha\beta$, with no need to find the roots themselves.

05 FORMING AN EQUATION FROM ITS ROOTS

A·05

The quadratic equation whose roots are α and β is

KEY

$$x^2-(\alpha+\beta)\,x+\alpha\beta=0$$

that is, $x^2-(\text{sum of roots})\,x+(\text{product of roots})=0$. Any non-zero multiple $k[x^2-(\alpha+\beta)x+\alpha\beta]=0$ has the same roots.

CUBIC VERSION

For roots α, β, γ the equation is

$$x^3-\left(\sum\alpha\right)x^2+\left(\sum\alpha\beta\right)x-\alpha\beta\gamma=0$$

B WORKING WITH THE ROOTS 06-09

06 SYMMETRIC FUNCTIONS OF THE ROOTS

B·06

Write $S=\alpha+\beta=-\frac{b}{a}$ and $P=\alpha\beta=\frac{c}{a}$. Every symmetric expression in the roots can be written in terms of S and P:

$$\alpha^2+\beta^2=S^2-2P$$

KEY

$$(\alpha-\beta)^2=S^2-4P=\frac{D}{a^2}$$

$$\alpha^3+\beta^3=S^3-3PS$$

$$\alpha^3-\beta^3=(\alpha-\beta)(S^2-P)$$

$$\alpha^4+\beta^4=\left(S^2-2P\right)^2-2P^2$$

$$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{S}{P}, \qquad \frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{S^2-2P}{P}$$

$$\alpha^2\beta+\alpha\beta^2=PS, \qquad \frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{S^2-2P}{P^2}$$

> **Higher powers.** For $\alpha^k+\beta^k$ with large k, do not expand. Use Newton's formula, section **19**, which steps up one power at a time from $S_0=2$ and $S_1=S$.

07 TRANSFORMATION OF EQUATIONS

B·07

Let α, β be the roots of $ax^2+bx+c=0$. The equation with the transformed roots is obtained by substituting for x:

NEW ROOTS	EQUATION	SUBSTITUTE
$-\alpha,\ -\beta$	$ax^2-bx+c=0$	$x\to -x$
$\frac{1}{\alpha},\ \frac{1}{\beta}$	$cx^2+bx+a=0$	$x\to\frac{1}{x}$
$k\alpha,\ k\beta$	$ax^2+kbx+k^2c=0$	$x\to\frac{x}{k}$
$\alpha+k,\ \beta+k$	$a(x-k)^2+b(x-k)+c=0$	$x\to x-k$
$\alpha^2,\ \beta^2$	$a^2x^2-(b^2-2ac)x+c^2=0$	$x\to\sqrt{x}$

> **General method.** To find the equation whose roots are $f(\alpha)$ and $f(\beta)$, put $y=f(x)$, solve for x in terms of y, substitute into the original equation and clear the result. This works for any invertible f.

08 COMMON ROOTS

B·08

Consider $a_1x^2+b_1x+c_1=0$ and $a_2x^2+b_2x+c_2=0$.

A. EXACTLY ONE COMMON ROOT

If α is the common root, solving the two equations simultaneously gives

KEY

$$\alpha=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$$

and eliminating α gives the condition

$$\left(c_1a_2-c_2a_1\right)^2=(b_1c_2-b_2c_1)(a_1b_2-a_2b_1)$$

B. BOTH ROOTS COMMON

The two equations must be proportional:

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$$

> **Note.** If both equations have rational coefficients and share one irrational root, then they share the conjugate as well, so both roots are common. The same applies to a shared non-real root of two real equations.

09 EQUATIONS REDUCIBLE TO A QUADRATIC

B·09

- i. **Biquadratic.** $ax^4+bx^2+c=0$: put $x^2=t$, solve for t, then take square roots.
- ii. **Reciprocal.** $ax^4+bx^3+cx^2+bx+a=0$: divide by x^2 and put $x+\frac{1}{x}=t$, using $x^2+\frac{1}{x^2}=t^2-2$.
- iii. **Composite.** $a\,[f(x)]^2+b\,f(x)+c=0$: put $f(x)=t$.
- iv. **Exponential.** $a\,m^{2x}+b\,m^x+c=0$: put $m^x=t$ and keep only $t>0$.
- v. **Logarithmic.** A quadratic in $\log x$: put $\log x=t$, then return through the domain $x>0$.
- vi. **Irrational.** Substitute for the radical, then square. Squaring can create roots that do not satisfy the original equation.

> **Always check.** For irrational, logarithmic and exponential equations, every value obtained must be tested in the original equation and against its domain. Extraneous roots are the most common source of lost marks here.

C THE QUADRATIC EXPRESSION 10-14

10 GRAPH OF THE QUADRATIC EXPRESSION

C·10

The graph of $y=ax^2+bx+c$ is a parabola. Completing the square gives

$$y=a\left(x+\frac{b}{2a}\right)^2-\frac{D}{4a}$$

so the parabola has

KEY

$$\text{vertex } \left(-\frac{b}{2a},\ -\frac{D}{4a}\right), \qquad \text{axis } x=-\frac{b}{2a}$$

- i. It opens upwards when $a>0$ and downwards when $a<0$.
- ii. It cuts the y-axis at $(0, c)$.
- iii. It meets the x-axis exactly at the real roots: two points if $D>0$, one point of tangency if $D=0$, and no point if $D<0$.

This gives six standard pictures, one for each pairing of the sign of a with $D>0$, $D=0$ and $D<0$. Nearly every question on the sign, range or root location of a quadratic is one of these six pictures in disguise.

11 SIGN OF THE QUADRATIC EXPRESSION

C·11

Let $f(x)=ax^2+bx+c$ with a, b, c real and $a\neq 0$.

- i. $a>0$ and $D<0$: $f(x)>0$ for every real x.
- ii. $a<0$ and $D<0$: $f(x)<0$ for every real x.
- iii. $a>0$ and $D=0$: $f(x)\geq 0$ for every real x, with equality only at $x=-\frac{b}{2a}$.
- iv. $a<0$ and $D=0$: $f(x)\leq 0$ for every real x, with equality only at $x=-\frac{b}{2a}$.
- v. $D>0$ with roots $\alpha<\beta$: $f(x)$ has the same sign as a for $x<\alpha$ and for $x>\beta$, and the sign opposite to a for $\alpha<x<\beta$.

> **Remember it as.** A quadratic keeps the sign of a everywhere except strictly between its real roots, where it flips. If there are no real roots there is nothing to flip, so the sign of a holds throughout.

12 MAXIMUM AND MINIMUM VALUE

C·12

From the vertex form, $f(x)=ax^2+bx+c$ attains its extreme value at $x=-\frac{b}{2a}$, and that value is

KEY

$$f\left(-\frac{b}{2a}\right)=-\frac{D}{4a}=\frac{4ac-b^2}{4a}$$

- i. $a>0$: this is the minimum, and the range is $\left[-\frac{D}{4a},\ \infty\right)$.
- ii. $a<0$: this is the maximum, and the range is $\left(-\infty,\ -\frac{D}{4a}\right]$.

> **On a restricted interval.** Over $x\in[p,q]$ the extreme values occur at the vertex when $-\frac{b}{2a}\in[p,q]$, and otherwise at whichever endpoint is nearer the vertex. Always compare $f(p)$, $f(q)$ and, when it is admissible, $f\left(-\frac{b}{2a}\right)$.

13 RANGE OF A RATIONAL EXPRESSION

C·13

To find the set of values taken by

$$y=\frac{a_1x^2+b_1x+c_1}{a_2x^2+b_2x+c_2}$$

as x runs over the reals, clear the denominator and read the result as a quadratic in x:

KEY

$$(a_1-ya_2)x^2+(b_1-yb_2)x+(c_1-yc_2)=0$$

A real x exists precisely when this quadratic has a real root, so the range is the set of y satisfying

$$(b_1-yb_2)^2-4(a_1-ya_2)(c_1-yc_2)\ \geq\ 0$$

> **Two conditions to check.**
> (i) The denominator must not vanish for any real x, that is $b_2^2-4a_2c_2<0$. Otherwise the expression is undefined somewhere and the method fails.
> (ii) The value of y making $a_1-ya_2=0$ must be handled separately, since there the equation is linear rather than quadratic.

14 QUADRATIC INEQUALITIES AND THE WAVY CURVE

C·14

To solve an inequality built from factors, such as

$$\frac{(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots}{(x-b_1)^{n_1}(x-b_2)^{n_2}\cdots}\ >\ 0$$

- i. Factor completely and make every leading coefficient positive, reversing the inequality sign whenever you multiply by a negative number.
- ii. Mark all critical points, the zeros of the numerator and of the denominator, on the number line in increasing order.
- iii. Start the curve above the line to the right of the largest critical point.
- iv. At a point of odd multiplicity the curve crosses the axis; at a point of even multiplicity it touches the axis and returns to the same side.
- v. Read off the intervals where the curve lies on the required side.

> **Endpoints.** Zeros of the numerator are included for $\geq$ and $\leq$ but excluded for strict inequalities. Zeros of the denominator are excluded always, whatever the inequality sign.

D LOCATION OF ROOTS & HIGHER DEGREE 15-19

15 LOCATION OF THE ROOTS

D·15

Let $f(x)=ax^2+bx+c$ with $a>0$, real roots $\alpha\leq\beta$, and let k, $k_1<k_2$ be real numbers. If $a<0$, first multiply the whole equation by -1.

- i. **Both roots greater than** k:
  $D\geq 0,\ f(k)>0,\ -\frac{b}{2a}>k$
- ii. **Both roots less than** k:
  $D\geq 0,\ f(k)>0,\ -\frac{b}{2a}<k$
- iii. k **lies between the roots** $(\alpha<k<\beta)$:
  $f(k)<0$, which forces $D>0$ on its own
- iv. **Both roots in** (k_1, k_2):
  $D\geq 0,\ f(k_1)>0,\ f(k_2)>0,\ k_1<-\frac{b}{2a}<k_2$
- v. **Exactly one root in** (k_1, k_2):
  $f(k_1)\cdot f(k_2)<0$
- vi. k_1 **and** k_2 **both lie between the roots:**
  $f(k_1)<0$ and $f(k_2)<0$

> **Where these come from.** Each condition is just the parabola picture written down: the sign of f at a test point tells you which side of the roots that point is on, the vertex abscissa $-\frac{b}{2a}$ says where the turning point sits, and D decides whether real roots exist at all.

16 POLYNOMIAL EQUATIONS OF HIGHER DEGREE

D·16

For the equation $a_0x^n+a_1x^{n-1}+\cdots+a_n=0$ with $a_0\neq 0$ and roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, the elementary symmetric sums are

$$\sum\alpha_1=-\frac{a_1}{a_0}, \quad \sum\alpha_1\alpha_2=\frac{a_2}{a_0}, \quad \sum\alpha_1\alpha_2\alpha_3=-\frac{a_3}{a_0}$$

KEY

$$\alpha_1\alpha_2\cdots\alpha_n=(-1)^n\,\frac{a_n}{a_0}$$

CUBIC

For $ax^3+bx^2+cx+d=0$ with roots α, β, γ:

$$\alpha+\beta+\gamma=-\frac{b}{a}, \quad \alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}, \quad \alpha\beta\gamma=-\frac{d}{a}$$

BIQUADRATIC

For $ax^4+bx^3+cx^2+dx+e=0$ with roots $\alpha, \beta, \gamma, \delta$:

$$\sum\alpha=-\frac{b}{a}, \quad \sum\alpha\beta=\frac{c}{a}, \quad \sum\alpha\beta\gamma=-\frac{d}{a}, \quad \alpha\beta\gamma\delta=\frac{e}{a}$$

17 NATURE OF ROOTS OF POLYNOMIAL EQUATIONS

D·17

- i. Every polynomial equation of degree n has exactly n roots in the complex numbers, counted with multiplicity.
- ii. If the coefficients are real, non-real roots occur in conjugate pairs. Hence a real polynomial equation of odd degree has at least one real root.
- iii. If the coefficients are rational, irrational roots of the form $p+\sqrt{q}$ occur in conjugate pairs with $p-\sqrt{q}$.
- iv. **Remainder theorem.** The remainder on dividing $f(x)$ by $(x-a)$ is $f(a)$.
- v. **Factor theorem.** $(x-a)$ is a factor of $f(x)$ if and only if $f(a)=0$.
- vi. **Descartes' rule of signs.** The number of positive real roots does not exceed the number of sign changes in $f(x)$, and the number of negative real roots does not exceed the number of sign changes in $f(-x)$. In each case the shortfall is even.
- vii. If $f(p)$ and $f(q)$ have opposite signs, at least one real root lies between p and q.
- viii. Between two consecutive real roots of $f(x)=0$ there lies at least one real root of $f'(x)=0$.

18 QUICK CONDITIONS ON THE ROOTS

D·18

For $ax^2+bx+c=0$ with real coefficients and $a\neq 0$, roots α, β:

REQUIRED PROPERTY OF THE ROOTS	CONDITION ON THE COEFFICIENTS
Roots are equal	$D=0$
Roots are real	$D\geq 0$
Roots are of opposite sign	$\frac{c}{a}<0$
Both roots are positive	$D\geq 0,\ \frac{b}{a}<0,\ \frac{c}{a}>0$
Both roots are negative	$D\geq 0,\ \frac{b}{a}>0,\ \frac{c}{a}>0$
Equal in magnitude, opposite in sign	$b=0$
Roots are reciprocals of each other	$a=c$
One root is 1	$a+b+c=0$
One root is -1	$a-b+c=0$
One root is zero	$c=0$
Both roots are zero	$b=c=0$
One root is infinite	$a=0$
Both roots are infinite	$a=b=0$

> **Cube roots of unity.** The equation $x^2+x+1=0$ has roots ω and ω^2, the non-real cube roots of unity, with $1+\omega+\omega^2=0$ and $\omega^3=1$. The equation $x^2-x+1=0$ has roots $-\omega$ and $-\omega^2$.

19 NEWTON'S FORMULA FOR POWER SUMS

D·19

Write $S_k=\alpha^k+\beta^k$ for the sum of the kth powers of the roots of $ax^2+bx+c=0$. Each root satisfies the equation, so multiplying $a\alpha^2+b\alpha+c=0$ by α^{k-2}, doing the same for β, and adding gives Newton's formula:

KEY

$$aS_k+bS_{k-1}+cS_{k-2}=0, \qquad k\geq 2$$

It is a recurrence, so every power sum follows from the two starting values

$$S_0=2, \qquad S_1=-\frac{b}{a}$$

Running it twice, for instance, gives

$$S_2=\frac{b^2-2ac}{a^2}, \qquad S_3=\frac{-b^3+3abc}{a^3}$$

GENERAL POLYNOMIAL

For $a_0x^n+a_1x^{n-1}+\cdots+a_n=0$ with roots $\alpha_1, \ldots, \alpha_n$ and $S_k=\sum\alpha_i^k$, the same argument gives

$$a_0S_k+a_1S_{k-1}+\cdots+a_nS_{k-n}=0, \qquad k>n$$

For the first few power sums the recurrence runs off the end of the coefficients, and a correction term appears:

$$a_0S_k+a_1S_{k-1}+\cdots+a_{k-1}S_1+k\,a_k=0, \qquad 1\leq k\leq n$$

CUBIC

For $ax^3+bx^2+cx+d=0$ the formula unrolls to

$$aS_1+b=0$$

$$aS_2+bS_1+2c=0$$

$$aS_3+bS_2+cS_1+3d=0$$

$$aS_k+bS_{k-1}+cS_{k-2}+dS_{k-3}=0, \qquad k\geq 4$$

> **Note.** S_0 equals the number of roots, so $S_0=n$ in general and $S_0=2$ for a quadratic. Sums of negative powers come from applying the same formula to the reciprocal equation, which is $cx^2+bx+a=0$ in the quadratic case.