[REVISION TOOL · JEE MAIN + ADVANCED · CLASS 11]

CONCEPT MAP

COMPLEX NUMBERS

4 TOPICS · 18 CORE IDEAS · ONE SHEET

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SCAN TO PRACTICE

A IOTA & THE COMPLEX NUMBER 01–04

01 INTEGRAL POWERS OF IOTA (A·01)

Euler introduced the symbol i, called iota, for the square root of -1, with the defining property $i^2=-1$. He also called it the imaginary unit.

A. POSITIVE INTEGRAL POWERS

$$i=\sqrt{-1},\quad i^2=-1,\quad i^3=i^2\cdot i=-i,\quad i^4=(i^2)^2=1$$

The powers then repeat with period four, so for any integer k

KEY

$$i^{4k}=1,\quad i^{4k+1}=i,\quad i^{4k+2}=-1,\quad i^{4k+3}=-i$$

For $n>4$, the value of i^n is i^r, where r is the remainder when n is divided by 4.

B. NEGATIVE INTEGRAL POWERS

$$i^{-1}=\frac{1}{i}=\frac{i}{i^2}=-i,\qquad i^{-2}=\frac{1}{i^2}=-1$$

$$i^{-3}=\frac{1}{i^3}=\frac{i}{i^4}=i,\qquad i^{-4}=\frac{1}{i^4}=1$$

For $n>4$, $i^{-n}=\dfrac{1}{i^r}$, where r is the remainder when n is divided by 4. By convention $i^0=1$.

> **Caution.** The rule $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$ holds only when at least one of a and b is non-negative. Ignoring this gives the false chain $-1=i^2=\sqrt{-1}\sqrt{-1}=\sqrt{(-1)(-1)}=\sqrt{1}=1$.

02 IMAGINARY QUANTITIES (A·02)

The square root of a negative real number is called an imaginary quantity, or an imaginary number.

For example $\sqrt{-4}=2i$, $\sqrt{-7}=\sqrt{7}\,i$ *and* $\sqrt{-1}=i$.

Every imaginary quantity can therefore be written as a real multiple of i.

03 COMPLEX NUMBERS (A·03)

If a and b are real numbers, then a number of the form

KEY

$$z=a+ib$$

is called a complex number. Here a is the real part of z and b is the imaginary part, written

$$a=\mathrm{Re}(z),\qquad b=\mathrm{Im}(z)$$

i. z is purely real if its imaginary part is zero, that is $\mathrm{Im}(z)=0$.
ii. z is purely imaginary if its real part is zero, that is $\mathrm{Re}(z)=0$.

The set of all complex numbers is denoted by $\mathbb{C}$, and every real number a is the complex number $a+i0$.

> **Note.** The order relations $<$ and $>$ are not defined on $\mathbb{C}$. Two complex numbers can be compared only for equality, never for size, unless both happen to be real.

04 EQUALITY OF COMPLEX NUMBERS (A·04)

Two complex numbers $z_1=a_1+ib_1$ and $z_2=a_2+ib_2$ are equal when their real parts are equal and their imaginary parts are equal:

$$z_1=z_2 \iff a_1=a_2 \text{ and } b_1=b_2$$

Equivalently, $\mathrm{Re}(z_1)=\mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1)=\mathrm{Im}(z_2)$.

> **Use.** One complex equation carries two real equations. Comparing real and imaginary parts is the standard way to turn a problem in $\mathbb{C}$ into a pair of simultaneous real equations.

B ALGEBRA OF COMPLEX NUMBERS 05–08

05 ADDITION AND SUBTRACTION (B·05)

For $z_1=a_1+ib_1$ and $z_2=a_2+ib_2$, addition is defined componentwise:

KEY

$$z_1+z_2=(a_1+a_2)+i(b_1+b_2)$$

so $\mathrm{Re}(z_1+z_2)=\mathrm{Re}(z_1)+\mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1+z_2)=\mathrm{Im}(z_1)+\mathrm{Im}(z_2)$.

PROPERTIES OF ADDITION

i. **Commutative.** $z_1+z_2=z_2+z_1$ for any two complex numbers.
ii. **Associative.** $(z_1+z_2)+z_3=z_1+(z_2+z_3)$ for any three.
iii. **Additive identity.** $0=0+i0$ satisfies $z+0=0+z=z$ for every $z\in\mathbb{C}$.
iv. **Additive inverse.** For every z there is $-z$ with $z+(-z)=(-z)+z=0$.

SUBTRACTION

Subtraction is addition of the additive inverse:

$$z_1-z_2=z_1+(-z_2)=(a_1-a_2)+i(b_1-b_2)$$

06 MULTIPLICATION (B·06)

Multiplying $z_1=a_1+ib_1$ by $z_2=a_2+ib_2$ and using $i^2=-1$ gives

KEY

$$z_1z_2=(a_1a_2-b_1b_2)+i(a_1b_2+a_2b_1)$$

PROPERTIES OF MULTIPLICATION

i. **Commutative.** $z_1z_2=z_2z_1$.
ii. **Associative.** $(z_1z_2)z_3=z_1(z_2z_3)$.
iii. **Multiplicative identity.** $1=1+i0$ satisfies $z\cdot1=1\cdot z=z$.
iv. **Multiplicative inverse.** Every non-zero $z=a+ib$ has an inverse z^{-1} with $z\,z^{-1}=1$, namely $z^{-1}=\dfrac{a}{a^2+b^2}+i\,\dfrac{-b}{a^2+b^2}$
v. **Distributive over addition.** $z_1(z_2+z_3)=z_1z_2+z_1z_3$.

07 DIVISION AND RECIPROCAL (B·07)

Division by a non-zero z_2 is multiplication by its inverse:

$$\frac{z_1}{z_2}=z_1\,z_2^{-1}=z_1\left(\frac{1}{z_2}\right),\qquad z_2\neq0$$

The reciprocal of a non-zero z is obtained by multiplying above and below by the conjugate:

KEY

$$\frac{1}{z}=\frac{\bar z}{z\bar z}=\frac{\bar z}{|z|^2}$$

> **In practice.** To simplify $\frac{z_1}{z_2}$, multiply numerator and denominator by $\bar z_2$. The denominator becomes the real number $|z_2|^2$, and the real and imaginary parts can then be read off directly.

08 CONJUGATE OF A COMPLEX NUMBER (B·08)

The conjugate of $z=a+ib$ is obtained by changing the sign of the imaginary part:

KEY

$$\bar z=a-ib$$

PROPERTIES OF THE CONJUGATE

i. $\overline{(\bar z)}=z$
ii. $z+\bar z=2\,\mathrm{Re}(z)$
iii. $z-\bar z=2i\,\mathrm{Im}(z)$
iv. $z=\bar z \iff z$ is purely real
v. $z+\bar z=0 \iff z$ is purely imaginary
vi. $z\bar z=\left[\mathrm{Re}(z)\right]^2+\left[\mathrm{Im}(z)\right]^2=|z|^2$
vii. $\overline{z_1+z_2}=\bar z_1+\bar z_2$ and $\overline{z_1-z_2}=\bar z_1-\bar z_2$
viii. $\overline{z_1z_2}=\bar z_1\,\bar z_2$
ix. $\overline{\left(\dfrac{z_1}{z_2}\right)}=\dfrac{\bar z_1}{\bar z_2},\quad z_2\neq0$

C MODULUS & ARGUMENT 09–13

09 MODULUS OF A COMPLEX NUMBER (C·09)

The modulus of $z=a+ib$ is denoted by $|z|$ and defined as

KEY

$$|z|=\sqrt{a^2+b^2}=\sqrt{\left[\mathrm{Re}(z)\right]^2+\left[\mathrm{Im}(z)\right]^2}$$

Geometrically $|z|$ is the distance of the point z from the origin in the Argand plane, so it is always a non-negative real number.

10 PROPERTIES OF THE MODULUS (C·10)

For $z,z_1,z_2\in\mathbb{C}$:

i. $|z|=0 \iff z=0$, that is $\mathrm{Re}(z)=\mathrm{Im}(z)=0$
ii. $|z|=|\bar z|=|-z|$
iii. $-|z|\le\mathrm{Re}(z)\le|z|$ and $-|z|\le\mathrm{Im}(z)\le|z|$
iv. $z\bar z=|z|^2$
v. $|z_1z_2|=|z_1||z_2|$
vi. $\left|\dfrac{z_1}{z_2}\right|=\dfrac{|z_1|}{|z_2|},\quad z_2\neq0$
vii. $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2\,\mathrm{Re}(z_1\bar z_2)$
viii. $|z_1-z_2|^2=|z_1|^2+|z_2|^2-2\,\mathrm{Re}(z_1\bar z_2)$
ix. $|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2)$
x. $|az_1-bz_2|^2+|bz_1+az_2|^2=(a^2+b^2)(|z_1|^2+|z_2|^2)$

> **Triangle inequality.** For all z_1,z_2,
> $\big||z_1|-|z_2|\big|\le|z_1+z_2|\le|z_1|+|z_2|$
> with equality on the right when z_1 and z_2 have the same argument.

11 ARGUMENT OF A COMPLEX NUMBER (C·11)

The angle θ that the vector representing z makes with the positive direction of the x-axis, measured anticlockwise, is called the argument or amplitude of z, written $\arg(z)$ or $\mathrm{amp}(z)$. From the right triangle in the Argand plane,

KEY

$$\tan\theta=\frac{y}{x}=\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\ \Rightarrow\ \theta=\tan^{-1}\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right)$$

This angle has infinitely many values differing by multiples of 2π. The unique value satisfying $-\pi<\theta\le\pi$ is the principal value of the argument, or the principal argument.

ALGORITHM FOR THE PRINCIPAL ARGUMENT

i. Find the acute angle $\alpha=\left|\tan^{-1}\left|\dfrac{y}{x}\right|\right|$, which lies between 0 and $\dfrac{\pi}{2}$.
ii. Locate the quadrant of the point $P(x,y)$ and read off the argument:
first quadrant: $\arg(z)=\alpha$
second quadrant: $\arg(z)=\pi-\alpha$
third quadrant: $\arg(z)=\alpha-\pi$
fourth quadrant: $\arg(z)=-\alpha$

> **Other convention.** If the argument is instead taken in $[0,2\pi)$, add 2π to the negative answers: the third quadrant gives $\pi+\alpha$ and the fourth gives $2\pi-\alpha$. Always state which convention is in use.

12 POLAR AND EULER FORM (C·12)

Writing $r=|z|$ and $\theta=\arg(z)$, the point $z=a+ib$ has $a=r\cos\theta$ and $b=r\sin\theta$, so

KEY

$$z=r(\cos\theta+i\sin\theta)=r\,e^{i\theta}$$

the polar form and the Euler form of z, using Euler's relation

$$e^{i\theta}=\cos\theta+i\sin\theta$$

Multiplication and division are especially simple in this form: moduli multiply and divide while arguments add and subtract.

$$z_1z_2=r_1r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)]$$

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)]$$

13 RESULTS ON MODULUS AND ARGUMENT (C·13)

Let z,z_1,z_2 be complex numbers with arguments θ_1 and θ_2 respectively. Then:

i. $\arg(\bar z)=-\arg(z)$
ii. $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)$
iii. $\arg(z_1\bar z_2)=\arg(z_1)-\arg(z_2)$
iv. $\arg\left(\dfrac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)$
v. $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\theta_1-\theta_2)$
vi. $|z_1-z_2|^2=|z_1|^2+|z_2|^2-2|z_1||z_2|\cos(\theta_1-\theta_2)$
vii. $|z_1+z_2|=|z_1-z_2| \iff \arg(z_1)-\arg(z_2)=\dfrac{\pi}{2}$
viii. $|z_1+z_2|=|z_1|+|z_2| \iff \arg(z_1)=\arg(z_2)$

> **Note.** The four argument identities hold up to an added multiple of 2π. If principal values are demanded, adjust the answer back into $(-\pi,\pi]$ at the end.

D POWERS & ROOTS 14–18

14 SQUARE ROOTS OF A COMPLEX NUMBER (D·14)

For a complex number z with $\mathrm{Im}(z)>0$,

KEY

$$\sqrt{z}=\pm\left[\sqrt{\tfrac12\{|z|+\mathrm{Re}(z)\}}\ +\ i\sqrt{\tfrac12\{|z|-\mathrm{Re}(z)\}}\right]$$

and for $\mathrm{Im}(z)<0$,

$$\sqrt{z}=\pm\left[\sqrt{\tfrac12\{|z|+\mathrm{Re}(z)\}}\ -\ i\sqrt{\tfrac12\{|z|-\mathrm{Re}(z)\}}\right]$$

> **Reading the sign.** The two expressions differ only in the sign before i, and that sign follows the sign of $\mathrm{Im}(z)$. Both square roots always come as a $\pm$ pair.

15 DE MOIVRE'S THEOREM (D·15)

For every integer n,

KEY

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$$

and consequently

$$(\cos\theta-i\sin\theta)^n=\cos n\theta-i\sin n\theta$$

$$\frac{1}{\cos\theta+i\sin\theta}=(\cos\theta+i\sin\theta)^{-1}=\cos\theta-i\sin\theta$$

> **Caution.** The theorem does not apply to $\sin\theta+i\cos\theta$, so $(\sin\theta+i\cos\theta)^n\neq\sin n\theta+i\cos n\theta$
> Convert first, using $\sin\theta+i\cos\theta=i(\cos\theta-i\sin\theta)$, which gives $(\sin\theta+i\cos\theta)^n=i^n(\cos n\theta-i\sin n\theta)$.

> **Fractional index.** If $n=\dfrac{p}{q}$ in lowest terms, then $(\cos\theta+i\sin\theta)^{p/q}$ has q distinct values, one of which is $\cos\dfrac{p\theta}{q}+i\sin\dfrac{p\theta}{q}$.

16 ROOTS OF A COMPLEX NUMBER (D·16)

Let $z=a+ib$ have modulus r and argument θ. The nth roots of z are

KEY

$$z^{1/n}=r^{1/n}\left\{\cos\frac{\theta+2k\pi}{n}+i\sin\frac{\theta+2k\pi}{n}\right\}$$

for $k=0,1,2,\dots,n-1$, which gives exactly n distinct values. Taking $k=0$ gives the principal root

$$r^{1/n}\left\{\cos\frac{\theta}{n}+i\sin\frac{\theta}{n}\right\}$$

> **Geometry.** All n roots have the same modulus $r^{1/n}$, so they lie on a circle of that radius centred at the origin, spaced $\dfrac{2\pi}{n}$ apart. They are the vertices of a regular polygon of n sides.

17 CUBE ROOTS OF UNITY (D·17)

Putting $z=(1)^{1/3}$ gives $z=e^{i2k\pi/3}$, so the cube roots of unity are $1,\ \alpha,\ \alpha^2$ with $\alpha=e^{i2\pi/3}$. Writing ω for $e^{i2\pi/3}$,

KEY

$$\omega=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}=-\frac12+i\frac{\sqrt3}{2}$$

$$\omega^2=-\frac12-i\frac{\sqrt3}{2}$$

PROPERTIES OF THE CUBE ROOTS OF UNITY

i. The cube roots of unity are $1,\ \omega,\ \omega^2$.
ii. $\arg(\omega)=\dfrac{2\pi}{3}$ and $\arg(\omega^2)=\dfrac{4\pi}{3}$
iii. $z^3-1=(z-1)(z-\omega)(z-\omega^2)$
iv. ω and ω^2 are the roots of $z^2+z+1=0$
v. They lie on the unit circle $|z|=1$ and divide its circumference into three equal parts.
vi. $1+\omega^n+\omega^{2n}=\begin{cases}0, & n \text{ not a multiple of } 3\\ 3, & n \text{ a multiple of } 3\end{cases}$
vii. The cube roots of -1 are $-1,\ -\omega,\ -\omega^2$.
viii. $z^3+1=(z+1)(z+\omega)(z+\omega^2)$
ix. $-\omega$ and $-\omega^2$ are the roots of $z^2-z+1=0$
x. $a^3+b^3=(a+b)(a+b\omega)(a+b\omega^2)$ and
$a^3-b^3=(a-b)(a-b\omega)(a-b\omega^2)$
$a^2+b^2+c^2-ab-bc-ca=(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$
$a^3+b^3+c^3-3abc=(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$
xi. $1+\omega+\omega^2=0$
xii. $\omega^{3n}=1$ for every integer n

18 nTH ROOTS OF UNITY (D·18)

Let $z=1^{1/n}$. Since $1=\cos2r\pi+i\sin2r\pi$ for every integer r,

$$z=(\cos2r\pi+i\sin2r\pi)^{1/n}=\cos\frac{2r\pi}{n}+i\sin\frac{2r\pi}{n}=e^{i2r\pi/n}$$

for $r=0,1,2,\dots,n-1$. Writing $\alpha=e^{i2\pi/n}$, the nth roots of unity are

KEY

$$1\,(=\alpha^0),\ \alpha,\ \alpha^2,\ \alpha^3,\ \dots,\ \alpha^{n-1}$$

$$\alpha=e^{i2\pi/n}=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$$

PROPERTIES OF THE nTH ROOTS OF UNITY

i. They form a GP with common ratio $e^{i2\pi/n}$.
ii. Their sum is always zero.
iii. The sum of the pth powers of the nth roots of unity is zero when p is not a multiple of n.
iv. That sum equals n when p is a multiple of n.
v. Their product is $(-1)^{n-1}$.
vi. They lie on the unit circle $|z|=1$ and divide its circumference into n equal parts.