CONCEPT MAP

RELATIONS AND FUNCTIONS

4 TOPICS · 19 CORE IDEAS · ONE SHEET

SCAN TO PRACTICE

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A RELATIONS 01–06

01 ORDERED PAIRS AND CARTESIAN PRODUCT A·01

A. ORDERED PAIR

A pair of elements written in a definite order is an ordered pair. In (x, y) the element x is the first component, or first coordinate, and y is the second.

B. CARTESIAN PRODUCT

For sets A and B, the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$ is the cartesian product of A and B, read as A cross B:

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

For n sets $A_1, A_2, \ldots, A_n$ the product is the set of n-tuples

KEY

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_i \in A_i \text{ for } 1 \le i \le n\}$$

C. COUNTING

If A and B are finite, then $n(A \times B) = n(A) \cdot n(B)$, and more generally

$$n(A_1 \times A_2 \times \cdots \times A_m) = n(A_1)\, n(A_2) \cdots n(A_m)$$

02 PROPERTIES OF THE CARTESIAN PRODUCT A·02

Let A, B, C and D be sets and let ϕ be the empty set.

- i. $A \times B = \phi \iff A = \phi$ or $B = \phi$
- ii. If one of A and B is infinite and the other is non-empty, then $A \times B$ is infinite.
- iii. $A \times B = B \times A \iff A = B$

THEOREM ON THE CARTESIAN PRODUCT

- i. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- ii. $(A \cup B) \times C = (A \times C) \cup (B \times C)$
- iii. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- iv. $(A \cap B) \times C = (A \times C) \cap (B \times C)$
- v. $(A \cup B) \times (C \cup D) = (A \times C) \cup (A \times D) \cup (B \times C) \cup (B \times D)$
- vi. $(A - B) \times C = (A \times C) - (B \times C)$
- vii. $A \times (B - C) = A \times B - A \times C$

03 RELATION, DOMAIN AND RANGE A·03

A relation R from a non-empty set A to a non-empty set B is a subset of the cartesian product:

KEY

$$R \subseteq A \times B$$

If $(a, b) \in R$ we say a is related to b and write $a\,R\,b$.

DOMAIN AND RANGE

- i. The set of all first components of the ordered pairs in R is the domain of R, written Dom R.
- ii. The set of all second components is the range of R, written range(R).

NUMBER OF RELATIONS

If A and B are finite with $n(A) = m$ and $n(B) = n$, then the number of relations from A to B is

$$2^{mn}$$

which counts the empty relation and the whole of $A \times B$ among them.

Relation on a set. If A is a set, any subset of $A \times A$ is a binary relation on A, or simply a relation on A.

04 TYPES OF RELATIONS A·04

Let R be a relation on a non-empty set X.

- i. **Empty or void relation.** No element of X is related to any element of X, so $R = \phi \subseteq X \times X$.
- ii. **Universal relation.** Every element is related to every element, so $R = X \times X$.
- iii. **Reflexive.** $(x, x) \in R$ for all $x \in X$.
- iv. **Symmetric.** $(x, y) \in R \Rightarrow (y, x) \in R$.
- v. **Transitive.** $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$.
- vi. **Equivalence relation.** R is reflexive, symmetric and transitive at the same time.

Note. The empty and the universal relation are the two extreme relations on a set: one contains no pair at all, the other contains every pair.

05 COMPOSITION AND INVERSE OF RELATIONS A·05

A. COMPOSITION

Let R be a relation from A to B and S a relation from B to C. Their composition $S \circ R$ is a relation from A to C:

KEY

$$S \circ R = \{(a, c) \in A \times C : \exists\, b \in B \text{ with } (a, b) \in R \text{ and } (b, c) \in S\}$$

THEOREM ON COMPOSITION

- i. $S \circ R \neq \phi$ if and only if range$(R) \cap$ Dom$(S) \neq \phi$
- ii. Dom$(S \circ R) =$ Dom(R)
- iii. range$(S \circ R) \subseteq$ range(S)
- iv. For relations $R \subseteq A \times B$, $S \subseteq B \times C$ and $T \subseteq C \times D$, $(T \circ S) \circ R = T \circ (S \circ R)$

B. INVERSE RELATION

For a relation R from A to B, the inverse relation R^{-1} from B to A is

$$R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$$

THEOREM ON INVERSE RELATIONS

- i. $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$
- ii. $(R^{-1})^{-1} = R$

06 PARTITION AND EQUIVALENCE CLASSES A·06

PARTITION OF A SET

Let X be a non-empty set. A class of subsets of X is a partition of X if the subsets are pairwise disjoint and their union is the whole of X.

EQUIVALENCE CLASS

Let R be an equivalence relation on X. For $x \in X$, the set

$$R(x) = \{y \in X : (x, y) \in R\}$$

is the equivalence class of x with respect to R, also called the R-class of x.

Why it matters. The equivalence classes of an equivalence relation on X are pairwise disjoint and cover X, so every equivalence relation partitions the set, and every partition arises this way.

B FUNCTIONS & TYPES 07–11

07 FUNCTION, DOMAIN, CODOMAIN AND RANGE B·07

A relation f from a set A to a set B is a function from A to B if to each $a \in A$ there corresponds exactly one $b \in B$ with $(a, b) \in f$. We write

KEY

$$f : A \to B$$

A relation $f \subseteq A \times B$ is a function from A to B precisely when

- i. domain$(f) = A$, so every element of A is used, and
- ii. $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$, so no element of A has two images.

DOMAIN, CODOMAIN AND RANGE

For $f : A \to B$ the set A is the domain and B is the codomain. The range is the set of images

$$f(A) = \{f(a) : a \in A\}$$

also called the image set of A under f. Always $f(A) \subseteq B$.

08 REAL VALUED FUNCTIONS AND OPERATIONS B·08

If the range of a function is a subset of the real numbers $\mathbb{R}$, the function is called a real valued function.

OPERATIONS AMONG REAL VALUED FUNCTIONS

Let f and g be real valued functions defined on a set A. On A we define $f + g$, $f - g$ and $f \cdot g$ by

- i. $(f + g)(a) = f(a) + g(a)$
- ii. $(f - g)(a) = f(a) - g(a)$
- iii. $(f \cdot g)(a) = f(a) \cdot g(a)$
- iv. If $g(a) \neq 0$ for all $a \in A$, then $\left(\dfrac{f}{g}\right)(a) = \dfrac{f(a)}{g(a)}$
- v. If n is a positive integer, then $f^n(a) = (f(a))^n$

09 ONE-ONE FUNCTION (INJECTION) B·09

A function $f : A \to B$ is one-one, or an injection, if distinct elements of A have distinct images:

KEY

$$f(a_1) \neq f(a_2) \text{ for all } a_1 \neq a_2 \text{ in } A$$

Equivalently, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.

Test. If $f(x)$ is strictly increasing throughout its domain, or strictly decreasing throughout, then f is one-one. For a differentiable f this means $f'(x) > 0$ everywhere, or $f'(x) < 0$ everywhere.

10 ONTO FUNCTION AND BIJECTION B·10

A. ONTO FUNCTION (SURJECTION)

A function $f : A \to B$ is onto, or a surjection, if its range equals its codomain, that is

KEY

$$f(A) = B$$

so for each $b \in B$ there exists at least one $a \in A$ with $f(a) = b$.

B. ONE-ONE AND ONTO (BIJECTION)

A function $f : A \to B$ is a bijection if it is both an injection and a surjection, that is both one-one and onto.

Note. Only a bijection can be inverted. Being one-one alone, or onto alone, is not enough for f^{-1} to exist as a function.

11 COMPOSITION OF FUNCTIONS B·11

Let $f : A \to B$ and $g : B \to C$. The composition of f with g, written $g \circ f$, is the function

KEY

$$g \circ f : A \to C, \qquad (g \circ f)(a) = g(f(a)) \quad \forall\, a \in A$$

Composition is associative: for $f : A \to B$, $g : B \to C$ and $h : C \to D$,

$$(h \circ g) \circ f = h \circ (g \circ f)$$

PROPERTIES OF COMPOSITION

- i. If f and g are injections, then $g \circ f$ is an injection.
- ii. If $g \circ f$ is an injection, then f is an injection.
- iii. If f and g are surjections, then $g \circ f$ is a surjection.
- iv. If $g \circ f$ is a surjection, then g is a surjection.
- v. If f and g are bijections, then $g \circ f : A \to C$ is a bijection.

C SPECIAL FUNCTIONS 12–15

12 IDENTITY FUNCTION C·12

A function $f : A \to A$ is the identity function on A if it leaves every element unchanged:

KEY

$$f(x) = x \quad \forall\, x \in A$$

It is denoted by I_A, and it is a bijection from A onto itself.

13 INVERSE FUNCTION C·13

Let $f : A \to B$ be a bijection. Then there exists a function $g : B \to A$ satisfying

KEY

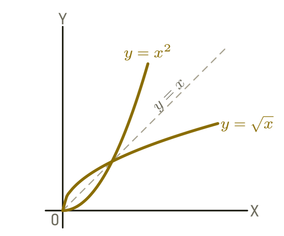
$$g \circ f = I_A, \qquad f \circ g = I_B$$

that is, $g(f(a)) = a$ for every $a \in A$ and $f(g(b)) = b$ for every $b \in B$. This g is the inverse of f, written f^{-1}.

PROPERTIES OF THE INVERSE

- i. The inverse of a bijection is unique.
- ii. The graphs of f and f^{-1} are mirror images of each other in the line $y = x$.

For instance $y = x^2$ on $x \ge 0$ and $y = \sqrt{x}$ are inverse to each other:



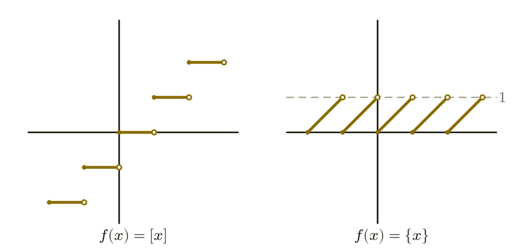
14 INTEGRAL AND FRACTIONAL PART FUNCTIONS C·14

For a real number x, the greatest integer less than or equal to x is the integral part of x, written $[x]$, and the integral part function is $f(x) = [x]$. What is left over,

KEY

$$\{x\} = x - [x]$$

is the fractional part of x, and $f(x) = \{x\}$ is the fractional part function, which always takes values in $[0, 1)$.



STANDARD RESULTS

- i. $[x + y] = \begin{cases} [x] + [y], & \{x\} + \{y\} < 1 \\ [x] + [y] + 1, & \{x\} + \{y\} \ge 1 \end{cases}$
- ii. $[x + y] \ge [x] + [y]$, with equality if and only if $\{x\} + \{y\} < 1$.
- iii. If x or y is an integer, then $[x + y] = [x] + [y]$.

15 PERIODIC FUNCTION C·15

Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. A positive real number T is a period of f if

KEY

$$f(x + T) = f(x)$$

whenever both x and $x + T$ belong to A. A function with such a T is called periodic, and the smallest such T, when it exists, is the fundamental period.

D BEHAVIOUR OF FUNCTIONS 16–19

16 INCREASING AND DECREASING FUNCTIONS D·16

INCREASING

f is increasing if the output rises as the input rises: larger x gives larger $f(x)$. For a differentiable function this holds when

KEY

$$f'(x) > 0$$

DECREASING

f is decreasing if the output falls as the input rises: larger x gives smaller $f(x)$. For a differentiable function this holds when

$$f'(x) < 0$$

Link to injectivity. A function that is strictly increasing on its whole domain, or strictly decreasing on its whole domain, is automatically one-one, and so is invertible onto its range.

17 EVEN AND ODD FUNCTIONS D·17

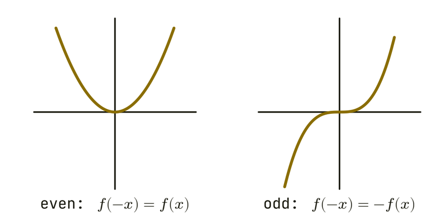
Let $f : A \to \mathbb{R}$ with A symmetric about the origin. Then f is even if

KEY

$$f(-x) = f(x) \quad \forall\, x \in A$$

and f is odd if

$$f(-x) = -f(x) \quad \forall\, x \in A$$



18 PROPERTIES OF ODD AND EVEN FUNCTIONS D·18

- i. $f(x) - f(-x) = 0 \Rightarrow f(x)$ is even.
- ii. $f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd.
- iii. A function need not be either even or odd.
- iv. The inverse of an even function is not defined, because an even function is never one-one on a symmetric domain.
- v. The graph of an even function is symmetric about the Y-axis; the graph of an odd function is symmetric about the origin.
- vi. $f(x) = 0$ is the only function that is both even and odd.

Every function can be split into an even part and an odd part:

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{even}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{odd}}$$

19 GENERAL FUNCTIONAL EQUATIONS D·19

Let x and y be independent variables. Each functional equation below forces the shape of f:

- i. $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$ or $f(x) = 0$
- ii. $f(xy) = f(x)\,f(y) \Rightarrow f(x) = x^n,\ n \in \mathbb{R}$
- iii. $f(x + y) = f(x)\,f(y) \Rightarrow f(x) = a^{kx}$
- iv. $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is any constant

Method. Substitute convenient values such as $x = y = 1$, $x = y = 0$ or $y = \dfrac{1}{x}$ to pin down $f(1)$ and $f(0)$ first, then look for the pattern above.