CONCEPT MAP

LIMIT, CONTINUITY AND DIFFERENTIABILITY

4 TOPICS · 20 CORE IDEAS · ONE SHEET

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SCAN TO PRACTICE

A LIMITS & THEIR ALGEBRA 01–04

01 LEFT HAND AND RIGHT HAND LIMITS
A · 01

For a function $f(x)$ at $x = a$, the two one-sided limits are

KEY

$$\text{LHL} = \lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h), \qquad \text{RHL} = \lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h)$$

where $h > 0$ in both. If LHL = RHL, then $\lim_{x\to a} f(x)$ exists and equals their common value.

> **When the limit fails.** If the two one-sided limits exist but are unequal, or if either of them fails to exist, then $\lim_{x\to a} f(x)$ does not exist. The value $f(a)$ plays no part in this: a limit is about the approach, not the arrival.

02 ALGEBRA OF LIMITS
A · 02

Let $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$, with l and m both existing. Then:

- **i.** $\lim_{x\to a}(f \pm g)(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = l \pm m$
- **ii.** $\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = lm$
- **iii.** $\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{l}{m}$, provided $m \neq 0$
- **iv.** $\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x)$, where k is a constant
- **v.** $\lim_{x\to a} |f(x)| = \left|\lim_{x\to a} f(x)\right| = |l|$
- **vi.** $\lim_{x\to a}\{f(x)\}^{g(x)} = l^m$
- **vii.** $\lim_{x\to a}(f\circ g)(x) = f\left(\lim_{x\to a} g(x)\right) = f(m)$

Two important cases of the last result:

$$\lim_{x\to a} \log f(x) = \log\left(\lim_{x\to a} f(x)\right) = \log l, \qquad \lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)} = e^l$$

03 ORDER RESULTS AND THE SANDWICH THEOREM
A · 03

If $f(x) \leq g(x)$ for every x in a deleted neighbourhood of a, then the limits keep the same order:

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

SANDWICH THEOREM

If $f(x) \leq g(x) \leq h(x)$ for every x in a deleted neighbourhood of a, and the outer two functions share a limit, then the middle one is trapped:

KEY

$$\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x) \implies \lim_{x\to a} g(x) = l$$

Also, if a function grows without bound, its reciprocal vanishes:

$$\lim_{x\to a} f(x) = +\infty \text{ or } -\infty \implies \lim_{x\to a} \frac{1}{f(x)} = 0$$

04 INDETERMINATE FORMS
A · 04

The following seven forms carry no information on their own and are called indeterminate:

$$\frac{\infty}{\infty}, \quad \frac{0}{0}, \quad \infty - \infty, \quad 0\times\infty, \quad 1^\infty, \quad 0^0, \quad \infty^0$$

> **What they mean.** An indeterminate form is not a value, it is a signal that the limit needs more work: factorise, rationalise, use a standard result, or apply L'Hospital's rule. Forms such as $\frac{k}{0}$ with $k \neq 0$, or 0^∞, are not indeterminate.

B EVALUATION OF LIMITS 05–12

05 FACTORISATION METHOD
B · 05

Suppose that on putting $x = a$ the rational function $\frac{f(x)}{g(x)}$ takes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then $(x-a)$ is a factor of both $f(x)$ and $g(x)$:

KEY

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{(x-a)\,f_1(x)}{(x-a)\,g_1(x)} = \lim_{x\to a} \frac{f_1(x)}{g_1(x)}$$

Factorise numerator and denominator, cancel the common factor $(x-a)$, then put $x = a$ again. If the form repeats, repeat the process until a meaningful number appears.

06 RATIONALISATION METHOD
B · 06

This is used when the numerator or the denominator, or both, involve square roots. If an expression containing a square root vanishes when x is replaced by the value being approached, multiply above and below by its conjugate:

KEY

$$\frac{\sqrt{A}-\sqrt{B}}{x-a} \times \frac{\sqrt{A}+\sqrt{B}}{\sqrt{A}+\sqrt{B}} = \frac{A-B}{(x-a)(\sqrt{A}+\sqrt{B})}$$

The troublesome radical disappears from the numerator, and the vanishing factor can then be cancelled as before.

07 STANDARD ALGEBRAIC RESULT
B · 07

If an expression reduces to the form $\frac{x^n - a^n}{x-a}$, where $n \in \mathbb{Q}$, then it can be evaluated by

KEY

$$\lim_{x\to a} \frac{x^n - a^n}{x-a} = n\,a^{n-1}$$

> **Use.** This one formula replaces the whole factorisation for any rational index, positive or negative. It is also exactly the derivative of x^n at $x = a$, which is why it turns up so often.

08 LIMITS AT INFINITY
B · 08

To evaluate a limit as $x \to \infty$ or $x \to -\infty$:

- **i.** Write the expression as a rational function $\frac{f(x)}{g(x)}$, if it is not one already.
- **ii.** If k is the highest power of x occurring in numerator and denominator, divide every term of both by x^k.
- **iii.** Use $\lim_{x\to\infty} \frac{1}{x^n} = 0$ for $n > 0$.

For positive integers m, n and non-zero reals a_0, b_0 this gives the standard result

KEY

$$\lim_{x\to\infty} \frac{a_0x^m + a_1x^{m-1} + \cdots + a_m}{b_0x^n + b_1x^{n-1} + \cdots + b_n} = \begin{cases} \frac{a_0}{b_0}, & m = n \\ 0, & m < n \\ \infty, & m > n \text{ and } a_0b_0 > 0 \\ -\infty, & m > n \text{ and } a_0b_0 < 0 \end{cases}$$

09 TRIGONOMETRIC LIMITS
B · 09

The following are the standard results, all with x in radians:

KEY

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x\to 0} \frac{\tan x}{x} = 1$$

$$\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1, \qquad \lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1$$

$$\lim_{x\to 0} \frac{\sin x^\circ}{x} = \frac{\pi}{180}, \qquad \lim_{x\to 0} \cos x = 1$$

$$\lim_{x\to a} \frac{\sin(x-a)}{x-a} = 1, \qquad \lim_{x\to a} \frac{\tan(x-a)}{x-a} = 1$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}, \qquad \lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \frac{1}{2}$$

> **Degrees.** The result $\frac{\sin x}{x} \to 1$ holds only in radians. In degrees the limit is $\frac{\pi}{180}$, as the third line shows.

10 EXPONENTIAL AND LOGARITHMIC LIMITS
B · 10

KEY

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a, \qquad a > 0$$

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x\to 0} \frac{\log_e(1+x)}{x} = 1$$

$$\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \frac{1}{\log_e a}, \qquad a > 0,\ a \neq 1$$

11 LIMITS OF THE FORM 1 TO THE POWER INFINITY
B · 11

Theorem. If $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$, then

$$\lim_{x\to a} \{1 + f(x)\}^{1/g(x)} = e^{\lim_{x\to a} \frac{f(x)}{g(x)}}$$

Equivalently, if $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$, then

KEY

$$\lim_{x\to a}\{f(x)\}^{g(x)} = \lim_{x\to a} [1 + \{f(x) - 1\}]^{g(x)} = e^{\lim_{x\to a}\{f(x)-1\}\,g(x)}$$

PARTICULAR CASES

- **i.** $\lim_{x\to 0}(1+x)^{1/x} = e$
- **ii.** $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$
- **iii.** $\lim_{x\to 0}(1+\lambda x)^{1/x} = e^\lambda$
- **iv.** $\lim_{x\to\infty}\left(1+\frac{\lambda}{x}\right)^x = e^\lambda$

12 L'HOSPITAL'S RULE
B · 12

Let f and g be two functions such that

- **i.** $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$
- **ii.** both are continuous at $x = a$
- **iii.** both are differentiable at $x = a$
- **iv.** $f'(x)$ and $g'(x)$ are continuous at $x = a$

Then the limit of the ratio equals the limit of the ratio of the derivatives:

KEY

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

GENERALISATION

If $\frac{f'(x)}{g'(x)}$ again takes the form $\frac{0}{0}$ and f', g' themselves satisfy the same four conditions, apply the rule once more:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$$

The process may be repeated as many times as necessary until the limit is reached.

> **Remark.** The rule applies equally when $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$. It does not apply to any other indeterminate form directly: rewrite $0\times\infty$, $\infty - \infty$ or 1^∞ as a quotient first.

C CONTINUITY 13–17

13 CONTINUITY AT A POINT
C · 13

A function $f(x)$ is continuous at a point $x = a$ of its domain if and only if the limit exists and equals the value:

KEY

$$\lim_{x\to a} f(x) = f(a)$$

Written out in one-sided form, f is continuous at $x = a$ exactly when

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

ONE-SIDED CONTINUITY

- **i.** If $\lim_{x\to a^-} f(x) = f(a)$, then f is continuous from the left at $x = a$.
- **ii.** If $\lim_{x\to a^+} f(x) = f(a)$, then f is continuous from the right at $x = a$.

> **Three things must hold.** $f(a)$ must be defined, the limit must exist, and the two must agree. Continuity fails if any one of them fails.

14 CONTINUITY ON AN OPEN INTERVAL
C · 14

A function $f(x)$ is continuous on an open interval (a, b) if and only if it is continuous at every point of (a, b).

> **Why open is easy.** Every point of an open interval has room on both sides, so the ordinary two-sided definition applies at each one and no special treatment is needed.

15 CONTINUITY ON A CLOSED INTERVAL
C · 15

A function $f(x)$ is continuous on a closed interval $[a, b]$ if and only if all three of the following hold:

- **i.** f is continuous on the open interval (a, b)
- **ii.** $\lim_{x\to a^+} f(x) = f(a)$
- **iii.** $\lim_{x\to b^-} f(x) = f(b)$

In other words, f is continuous on $[a, b]$ if it is continuous on (a, b), continuous from the right at a, and continuous from the left at b.

16 DISCONTINUOUS FUNCTIONS
C · 16

A function f is discontinuous at a point a of its domain D if it is not continuous there. The point a is then a point of discontinuity of the function. The discontinuity may arise in any of the following ways:

- **i.** $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$, or both, may fail to exist.
- **ii.** Both one-sided limits may exist but be unequal.
- **iii.** Both may exist and be equal, yet their common value may not equal $f(a)$.

17 KINDS OF DISCONTINUITY
C · 17

REMOVABLE DISCONTINUITY

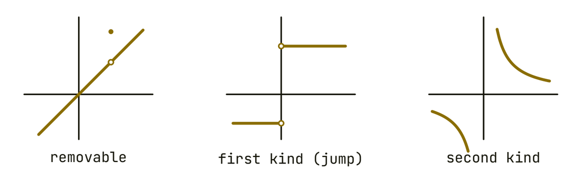
f has a removable discontinuity at $x = a$ if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$ but their common value is not equal to $f(a)$. It can be removed by assigning a suitable value to f at $x = a$.

DISCONTINUITY OF THE FIRST KIND

f has a discontinuity of the first kind at $x = a$ if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist but are not equal. It is of the first kind from the left at $x = a$ if the left limit exists but is not equal to $f(a)$, and from the right similarly.

DISCONTINUITY OF THE SECOND KIND

f has a discontinuity of the second kind at $x = a$ if neither one-sided limit exists. It is of the second kind from the left if the left limit does not exist, and from the right if the right limit does not exist.



removable · first kind (jump) · second kind

D DIFFERENTIABILITY 18–20

18 DIFFERENTIABILITY AT A POINT
D · 18

Let $f(x)$ be a real valued function defined on an open interval (a, b) and let $c \in (a, b)$. Then $f(x)$ is differentiable, or derivable, at $x = c$ if and only if

KEY

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} \quad \text{exists finitely}$$

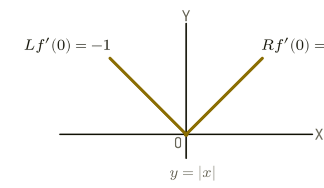
This limit is the derivative, or differential coefficient, of f at $x = c$, written $f'(c)$ or $\left(\frac{d}{dx}f(x)\right)_{x=c}$.

LEFT AND RIGHT HAND DERIVATIVES

$$Lf'(c) = \lim_{x\to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{h\to 0} \frac{f(c-h) - f(c)}{-h}$$

$$Rf'(c) = \lim_{x\to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$

So f is differentiable at $x = c$ if and only if $Lf'(c) = Rf'(c)$. If $Lf'(c) \neq Rf'(c)$, then f is not differentiable at $x = c$.



19 DIFFERENTIABILITY IN A SET
D · 19

A function $f(x)$ defined on $[a, b]$ is differentiable at the end points a and b if it is differentiable from the right at a and from the left at b, that is if

$$\lim_{x\to a^+} \frac{f(x) - f(a)}{x - a} \quad \text{and} \quad \lim_{x\to b^-} \frac{f(x) - f(b)}{x - b}$$

both exist. If f is derivable in the open interval (a, b) and also at the end points a and b, then f is derivable in the closed interval $[a, b]$.

A function is called a differentiable function if it is differentiable at every point of its domain. If f is differentiable at each $x \in \mathbb{R}$, it is said to be everywhere differentiable, as a constant function is.

20 STANDARD DIFFERENTIABILITY RESULTS
D · 20

- **i.** Every polynomial function is differentiable at each $x \in \mathbb{R}$.
- **ii.** The exponential function a^x, $a > 0$, is differentiable at each $x \in \mathbb{R}$.
- **iii.** Every constant function is differentiable at each $x \in \mathbb{R}$.
- **iv.** The logarithmic function is differentiable at each point of its domain.
- **v.** Trigonometric and inverse trigonometric functions are differentiable in their domains.
- **vi.** The sum, difference, product and quotient of two differentiable functions is differentiable.
- **vii.** The composition of differentiable functions is a differentiable function.

> **Differentiability implies continuity.** If f is differentiable at $x = c$, then f is continuous at $x = c$. The converse is false: $f(x) = |x|$ is continuous at $x = 0$ but has $Lf'(0) = -1$ and $Rf'(0) = 1$, so it is not differentiable there.