CONCEPT MAP

TRIGONOMETRY

5 TOPICS · 22 CORE IDEAS · ONE SHEET

SCAN TO PRACTICE

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A FUNCTIONS & COMPOUND ANGLES 01-04

01 DOMAIN AND RANGE OF TRIGONOMETRIC FUNCTIONS

A · 01

FUNCTION	DOMAIN	RANGE
$\sin\theta$	$\mathbb{R}$	$[-1,1]$
$\cos\theta$	$\mathbb{R}$	$[-1,1]$
$\tan\theta$	$\mathbb{R}-\left\{(2n+1)\frac{\pi}{2}:n\in\mathbb{Z}\right\}$	$\mathbb{R}$
$\cot\theta$	$\mathbb{R}-\{n\pi:n\in\mathbb{Z}\}$	$\mathbb{R}$
$\sec\theta$	$\mathbb{R}-\left\{(2n+1)\frac{\pi}{2}:n\in\mathbb{Z}\right\}$	$\mathbb{R}-(-1,1)$
$\operatorname{cosec}\theta$	$\mathbb{R}-\{n\pi:n\in\mathbb{Z}\}$	$\mathbb{R}-(-1,1)$

Reading the ranges. Sine and cosine are bounded, so their ranges are closed intervals. Secant and cosecant are their reciprocals, so their values never fall strictly between -1 and 1. Tangent and cotangent are unbounded and take every real value.

02 SUM AND DIFFERENCE FORMULAE

A · 02

KEY

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

$$\sin(A-B)=\sin A\cos B-\cos A\sin B$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$

$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B},\qquad \tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

$$\cot(A+B)=\frac{\cot A\cot B-1}{\cot B+\cot A},\qquad \cot(A-B)=\frac{\cot A\cot B+1}{\cot B-\cot A}$$

TWO USEFUL PRODUCTS

$$\sin(A+B)\sin(A-B)=\sin^2A-\sin^2B=\cos^2B-\cos^2A$$

$$\cos(A+B)\cos(A-B)=\cos^2A-\sin^2B=\cos^2B-\sin^2A$$

03 T-RATIOS OF MULTIPLE ANGLES

A · 03

KEY

$$\sin 2\theta=2\sin\theta\cos\theta=\frac{2\tan\theta}{1+\tan^2\theta}$$

$$\cos 2\theta=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1=1-2\sin^2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$1+\cos 2\theta=2\cos^2\theta,\qquad 1-\cos 2\theta=2\sin^2\theta$$

$$\frac{1+\cos 2\theta}{2}=\cos^2\theta,\qquad \frac{1-\cos 2\theta}{2}=\sin^2\theta$$

$$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$$

TRIPLE ANGLES

$$\sin 3\theta=3\sin\theta-4\sin^3\theta,\qquad \cos 3\theta=4\cos^3\theta-3\cos\theta$$

$$\tan 3\theta=\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$$

04 MAXIMUM AND MINIMUM VALUES

A · 04

For an expression of the form $a\sin x+b\cos x$, writing it as a single sine wave gives the extreme values

KEY

$$\text{maximum}=\sqrt{a^2+b^2},\qquad \text{minimum}=-\sqrt{a^2+b^2}$$

so the whole expression varies over $\left[-\sqrt{a^2+b^2},\ \sqrt{a^2+b^2}\right]$.

Where it comes from. Put $a=r\cos\phi$ and $b=r\sin\phi$ with $r=\sqrt{a^2+b^2}$. Then $a\sin x+b\cos x=r\sin(x+\phi)$, and $\sin(x+\phi)$ runs over $[-1,1]$.

B TRANSFORMATIONS 05-08

05 SUM AND DIFFERENCE INTO PRODUCT

B · 05

KEY

$$\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A-\sin B=2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

$$\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A-\cos B=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

TANGENTS AND COTANGENTS

$$\tan A+\tan B=\frac{\sin(A+B)}{\cos A\cos B},\qquad \tan A-\tan B=\frac{\sin(A-B)}{\cos A\cos B}$$

$$\cot A+\cot B=\frac{\sin(A+B)}{\sin A\sin B},\qquad \cot A-\cot B=\frac{\sin(B-A)}{\sin A\sin B}$$

06 PRODUCT INTO SUM OR DIFFERENCE

B · 06

KEY

$$2\sin A\cos B=\sin(A+B)+\sin(A-B)$$

$$2\cos A\sin B=\sin(A+B)-\sin(A-B)$$

$$2\cos A\cos B=\cos(A+B)+\cos(A-B)$$

$$2\sin A\sin B=\cos(A-B)-\cos(A+B)$$

Pairing. These four are exactly the four of section **05** read backwards. Going sum to product helps with factorising and solving equations; going product to sum helps with integration and with telescoping series.

07 T-RATIOS OF THE SUM OF THREE OR MORE ANGLES

B · 07

KEY

$$\sin(A+B+C)=\sin A\cos B\cos C+\cos A\sin B\cos C+\cos A\cos B\sin C-\sin A\sin B\sin C$$

$$\cos(A+B+C)=\cos A\cos B\cos C\times(1-\tan A\tan B-\tan B\tan C-\tan C\tan A)$$

$$\tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$$

GENERAL CASE

For n angles $A_1,A_2,\dots,A_n$,

$$\sin(A_1+A_2+\dots+A_n)=\cos A_1\cos A_2\cdots\cos A_n(S_1-S_3+S_5-S_7+\dots)$$

$$\cos(A_1+A_2+\dots+A_n)=\cos A_1\cos A_2\cdots\cos A_n(1-S_2+S_4-S_6+\dots)$$

$$\tan(A_1+A_2+\dots+A_n)=\frac{S_1-S_3+S_5-S_7+\dots}{1-S_2+S_4-S_6+\dots}$$

where $S_1=\sum\tan A_i$ is the sum of the separate tangents, $S_2=\sum\tan A_i\tan A_j$ the sum of products two at a time, S_3 the sum of products three at a time, and so on.

08 SPECIAL PRODUCT IDENTITIES

B · 08

Repeated doubling collapses a whole chain of cosines:

KEY

$$\cos A\cos 2A\cos 2^2A\cdots\cos 2^{n-1}A=\frac{\sin 2^nA}{2^n\sin A}$$

THE SIXTY DEGREE FAMILY

$$\sin\theta\,\sin(60^\circ-\theta)\,\sin(60^\circ+\theta)=\tfrac14\sin 3\theta$$

$$\cos\theta\,\cos(60^\circ-\theta)\,\cos(60^\circ+\theta)=\tfrac14\cos 3\theta$$

$$\tan\theta\,\tan(60^\circ-\theta)\,\tan(60^\circ+\theta)=\tan 3\theta$$

ARITHMETIC PROGRESSION OF ANGLES

$$\sin\alpha+\sin(\alpha+\beta)+\dots+\sin\{\alpha+(n-1)\beta\}=\frac{\sin\left\{\alpha+(n-1)\frac{\beta}{2}\right\}\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

$$\cos\alpha+\cos(\alpha+\beta)+\dots+\cos\{\alpha+(n-1)\beta\}=\frac{\cos\left\{\alpha+(n-1)\frac{\beta}{2}\right\}\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

C INVERSE FUNCTIONS: BASICS 09-14

09 DOMAIN AND RANGE OF INVERSE TRIGONOMETRIC FUNCTIONS

C · 09

FUNCTION	DOMAIN	RANGE
$\sin^{-1}x$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$\cos^{-1}x$	$[-1,1]$	$[0,\pi]$
$\tan^{-1}x$	$\mathbb{R}$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
$\cot^{-1}x$	$\mathbb{R}$	$(0,\pi)$
$\sec^{-1}x$	$\mathbb{R}-(-1,1)$	$[0,\pi]-\left\{\frac{\pi}{2}\right\}$
$\operatorname{cosec}^{-1}x$	$\mathbb{R}-(-1,1)$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$

Why the ranges are cut down. The trigonometric functions are not one-one on the whole line, so each is restricted to a principal branch before it is inverted. Every identity that follows holds only inside these ranges.

10 PROPERTY I: INVERSE OF A DIRECT FUNCTION

C · 10

KEY

$$\sin^{-1}(\sin\theta)=\theta,\qquad \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

$$\cos^{-1}(\cos\theta)=\theta,\qquad \theta\in[0,\pi]$$

$$\tan^{-1}(\tan\theta)=\theta,\qquad \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

$$\cot^{-1}(\cot\theta)=\theta,\qquad \theta\in(0,\pi)$$

$$\sec^{-1}(\sec\theta)=\theta,\qquad \theta\in[0,\pi]-\left\{\frac{\pi}{2}\right\}$$

$$\operatorname{cosec}^{-1}(\operatorname{cosec}\theta)=\theta,\qquad \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$$

Outside the range. If θ lies outside the stated interval the answer is not θ. Reduce θ into the principal branch first, for example $\sin^{-1}(\sin\frac{3\pi}{4})=\frac{\pi}{4}$, not $\frac{3\pi}{4}$.

11 PROPERTY II: DIRECT OF AN INVERSE FUNCTION

C · 11

KEY

$$\sin(\sin^{-1}x)=x,\qquad x\in[-1,1]$$

$$\cos(\cos^{-1}x)=x,\qquad x\in[-1,1]$$

$$\tan(\tan^{-1}x)=x,\qquad x\in\mathbb{R}$$

$$\cot(\cot^{-1}x)=x,\qquad x\in\mathbb{R}$$

$$\sec(\sec^{-1}x)=x,\qquad x\in\mathbb{R}-(-1,1)$$

$$\operatorname{cosec}(\operatorname{cosec}^{-1}x)=x,\qquad x\in\mathbb{R}-(-1,1)$$

This direction is safe. Unlike Property I, these hold for every x in the domain with no reduction needed, because the inverse already lands inside the principal branch.

12 PROPERTY III: NEGATIVE ARGUMENTS

C · 12

$$\sin^{-1}(-x)=-\sin^{-1}x,\qquad x\in[-1,1]$$

$$\cos^{-1}(-x)=\pi-\cos^{-1}x,\qquad x\in[-1,1]$$

$$\tan^{-1}(-x)=-\tan^{-1}x,\qquad x\in\mathbb{R}$$

$$\cot^{-1}(-x)=\pi-\cot^{-1}x,\qquad x\in\mathbb{R}$$

$$\sec^{-1}(-x)=\pi-\sec^{-1}x,\qquad x\in\mathbb{R}-(-1,1)$$

$$\operatorname{cosec}^{-1}(-x)=-\operatorname{cosec}^{-1}x,\qquad x\in\mathbb{R}-(-1,1)$$

Pattern. The three functions with a range symmetric about 0, namely $\sin^{-1}$, $\tan^{-1}$ and $\operatorname{cosec}^{-1}$, are odd. The three with range inside $[0,\pi]$ reflect through π instead.

13 PROPERTY IV: RECIPROCAL ARGUMENTS

C · 13

KEY

$$\sin^{-1}\left(\frac{1}{x}\right)=\operatorname{cosec}^{-1}x,\qquad x\in\mathbb{R}-(-1,1)$$

$$\cos^{-1}\left(\frac{1}{x}\right)=\sec^{-1}x,\qquad x\in\mathbb{R}-(-1,1)$$

$$\tan^{-1}\left(\frac{1}{x}\right)=\begin{cases}\cot^{-1}x, & x\in(0,\infty)\\ -\pi+\cot^{-1}x, & x\in(-\infty,0)\end{cases}$$

Mind the sign. Only the tangent case splits. For negative x the naive answer $\cot^{-1}x$ falls outside the range of $\tan^{-1}$, so π must be subtracted.

14 PROPERTY V: COMPLEMENTARY PAIRS

C · 14

KEY

$$\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2},\qquad x\in[-1,1]$$

$$\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2},\qquad x\in\mathbb{R}$$

$$\operatorname{cosec}^{-1}x+\sec^{-1}x=\frac{\pi}{2},\qquad x\in\mathbb{R}-(-1,1)$$

Use. These three turn any expression mixing a function with its co-function into one variable, which is usually the fastest route through an inverse-trigonometric equation.

D INVERSE FUNCTIONS: FURTHER 15-19

15 PROPERTY VI: DOUBLE ANGLE

D · 15

KEY

$$2\sin^{-1}x=\sin^{-1}\left(2x\sqrt{1-x^2}\right),\qquad -\frac{1}{\sqrt2}\le x\le\frac{1}{\sqrt2}$$

$$2\cos^{-1}x=\cos^{-1}(2x^2-1),\qquad 0\le x\le 1$$

$$2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right),\qquad -1<x<1$$

16 PROPERTY VII: TRIPLE ANGLE

D · 16

$$3\sin^{-1}x=\sin^{-1}(3x-4x^3),\qquad -\frac12\le x\le\frac12$$

$$3\cos^{-1}x=\cos^{-1}(4x^3-3x),\qquad \frac12\le x\le 1$$

$$3\tan^{-1}x=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right),\qquad -\frac{1}{\sqrt3}<x<\frac{1}{\sqrt3}$$

Ranges matter. The restrictions in Properties VI and VII are not decoration. Outside them the identity picks up a $\pm\pi$ or a sign change, which is the single most common source of error in this chapter.

17 PROPERTY VIII: CONVERTING A DOUBLE TANGENT

D · 17

KEY

$$2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right),\qquad -1\le x\le 1$$

$$2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right),\qquad 0\le x<\infty$$

Why they are useful. Together with $2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}$ from Property VI, these let a single substitution $x=\tan\theta$ clear the square roots from a whole family of expressions.

18 PROPERTY IX: SUM AND DIFFERENCE OF TWO INVERSE FUNCTIONS

D · 18

KEY

$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right),\qquad xy<1$$

$$\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right),\qquad xy>-1$$

$$\sin^{-1}x+\sin^{-1}y=\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}$$

for $-1\le x,y\le 1$ with $x^2+y^2\le 1$, and

$$\sin^{-1}x-\sin^{-1}y=\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}$$

for $-1\le x,y\le 1$ with $x^2+y^2\le 1$. For the cosine,

$$\cos^{-1}x+\cos^{-1}y=\cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\},\quad -1\le x,y\le 1,\ x+y\ge 0$$

$$\cos^{-1}x-\cos^{-1}y=\cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\},\quad -1\le x,y\le 1,\ x\le y$$

19 PROPERTY X: CONVERTING BETWEEN INVERSE FUNCTIONS

D · 19

Every inverse function can be rewritten in terms of any other. Taking $x>0$ in each line:

$$\sin^{-1}x=\cos^{-1}\left(\sqrt{1-x^2}\right)=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)=\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)=\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)=\operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

$$\cos^{-1}x=\sin^{-1}\left(\sqrt{1-x^2}\right)=\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)=\cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)=\sec^{-1}\left(\frac{1}{x}\right)=\operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$\tan^{-1}x=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)=\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)=\cot^{-1}\left(\frac{1}{x}\right)=\sec^{-1}\left(\sqrt{1+x^2}\right)=\operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$$

How to remember. Draw a right triangle with the given ratio marked, read the third side from Pythagoras, then read off whichever ratio is wanted. The formulas above are just that triangle written out.

E HEIGHTS & DISTANCES 20-22

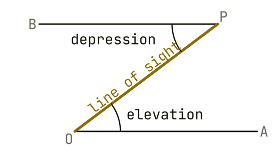
20 ANGLES OF ELEVATION AND DEPRESSION

E · 20

Let O and P be two points with P at the higher level, and let OA and PB be the horizontal lines through O and P.

i. If the observer is at O and P is the object, then OP is the line of sight and $\angle AOP$ is the **angle of elevation**.

ii. If the observer is at P and O is the object, then $\angle BPO$ is the **angle of depression**.



They are equal. The two horizontals are parallel, so the angle of elevation of P seen from O equals the angle of depression of O seen from P. Alternate angles do the work.

21 USEFUL RESULTS IN A TRIANGLE

E · 21

i. Any line perpendicular to a plane is perpendicular to every line lying in that plane.

ii. In a triangle, the internal bisector of an angle divides the opposite side in the ratio of the arms of that angle.

iii. In an isosceles triangle, the median to the base is perpendicular to the base.

iv. Angles in the same segment of a circle are equal.

Why they appear here. Height and distance problems are solved by finding one right triangle and one shared side. These four results are what let you find the right angle, or transfer a length from one triangle to the next.

22 THE M-N THEOREM

E · 22

Let D be a point on AB dividing it into $AD=m$ and $DB=n$, and let CD split the angle at C into α and β, meeting AB at an angle θ. Then

KEY

$$(m+n)\cot\theta=m\cot\alpha-n\cot\beta$$

and, in terms of the base angles A and B,

$$(m+n)\cot\theta=n\cot A-m\cot B$$