[REVISION TOOL · JEE MAIN + ADVANCED · CLASS 11]

CONCEPT MAP

STRAIGHT LINES

5 TOPICS · 28 CORE IDEAS · ONE SHEET

SCAN TO PRACTICE

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A POINTS IN THE PLANE 01-06

01 DISTANCE BETWEEN TWO POINTS
A·01

For $A(x_1, y_1)$ and $B(x_2, y_2)$,

KEY

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

IN POLAR COORDINATES

If the same two points are given as $A(r_1, \theta_1)$ and $B(r_2, \theta_2)$, then

$$AB^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)$$

Where it comes from. The polar version is just the cosine rule applied to $\triangle OAB$, with the angle at the origin equal to $\theta_1 - \theta_2$.

02 SECTION FORMULA
A·02

Let P divide the join of $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m : n$.

INTERNAL DIVISION

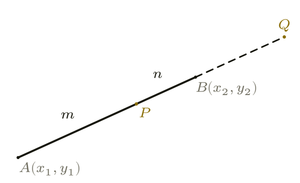
KEY

$$x = \frac{mx_2 + nx_1}{m+n}, \qquad y = \frac{my_2 + ny_1}{m+n}$$

EXTERNAL DIVISION

If P lies on AB produced, replace n by $-n$:

$$x = \frac{mx_2 - nx_1}{m-n}, \qquad y = \frac{my_2 - ny_1}{m-n}$$



TWO USEFUL SPECIAL CASES

i. Mid-point: put $m = n = 1$ to get $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

ii. The line $ax + by + c = 0$ divides AB in the ratio $-\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$; a positive value means the division is internal.

03 HARMONIC CONJUGATES
A·03

Four collinear points form a range. If P, Q, R, S lie on one line and the range (PQ, RS) has cross ratio -1, then R and S are called harmonic conjugates with respect to P and Q.

$$\frac{PR}{QR} \cdot \frac{SQ}{SP} = -1$$

Writing $\frac{PR}{QR} = \lambda$ makes this

KEY

$$\frac{PR}{QR} = \lambda, \qquad \frac{PS}{SQ} = -\lambda$$

What it says. R divides PQ internally and S divides the same segment externally, in the very same numerical ratio.

04 AREA OF A TRIANGLE
A·04

For vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3),

KEY

$$\Delta = \tfrac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

The same thing written as a sum of 2×2 determinants, which is far easier to write down from memory:

$$\Delta = \frac{1}{2}\left|\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix}\right|$$

Collinearity test. Three points are collinear exactly when this area is zero, i.e. when $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$.

05 AREA OF A POLYGON
A·05

For a polygon with vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ taken in order round the boundary, the area is

KEY

$$\frac{1}{2}\left|\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \cdots + \begin{vmatrix} x_n & x_1 \\ y_n & y_1 \end{vmatrix}\right|$$

Order matters. The vertices must be listed going round the polygon, all clockwise or all anticlockwise. Take them in any other order and the formula returns the wrong figure.

06 LOCUS OF A POINT
A·06

A locus is the path traced by a moving point that obeys one or more geometrical conditions. Circles, parabolas, ellipses and hyperbolas are all loci of this kind.

HOW TO FIND ONE

i. Let (h, k) be the coordinates of the moving point.

ii. Impose the given geometrical conditions on (h, k) and reduce them to a single relation between h and k.

iii. Replace h by x and k by y. The equation obtained is the required locus.

Do not skip step 2. Any auxiliary variable introduced along the way must be eliminated before the last step, or the result is not a locus at all.

B CENTRES OF A TRIANGLE 07-11

07 CENTROID
B·07

The medians of a triangle meet at the centroid. For vertices (x_1, y_1), (x_2, y_2), (x_3, y_3),

KEY

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Euler line. The orthocentre H, the centroid G and the circumcentre O are always collinear, and G divides HO in the ratio $2 : 1$.

08 INCENTRE
B·08

The internal bisectors of the angles meet at the incentre. With $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and the opposite sides $a = BC$, $b = CA$, $c = AB$,

KEY

$$I = \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

Match the weights. The weight attached to a vertex is the length of the side opposite that vertex, not the side through it. Getting this pairing wrong is the commonest slip in the whole topic.

09 EXCENTRES
B·09

The external bisectors of two angles and the internal bisector of the third meet at an excentre, so every triangle has three, one opposite each vertex. Each is obtained from the incentre by flipping the sign of one weight:

KEY

$$I_1 = \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$$

$$I_2 = \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}\right)$$

$$I_3 = \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}\right)$$

Here I_1, I_2, I_3 are the excentres opposite A, B, C respectively.

10 CIRCUMCENTRE
B·10

The perpendicular bisectors of the sides meet at the circumcentre, the one point equidistant from all three vertices.

Writing $OA = OB = OC$ gives two equations in the two unknown coordinates; solving them locates O. When the angles are known it is quicker to use $O = (x, y)$ with

KEY

$$x = \frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \quad y = \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$$

11 ORTHOCENTRE
B·11

The three altitudes meet at the orthocentre. It can be found by writing down any two altitudes and intersecting them, or directly from $H = (x, y)$ with

KEY

$$x = \frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \quad y = \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}$$

Right-angled triangle. The orthocentre is the vertex holding the right angle, and the circumcentre is the mid-point of the hypotenuse.

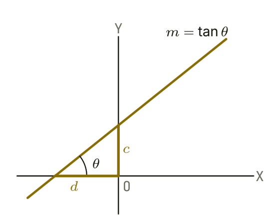
C EQUATION OF A LINE 12-18

12 SLOPE OF A LINE
C·12

If θ is the angle a line makes with the positive direction of the X-axis, its slope, or gradient, is

KEY

$$m = \tan\theta, \qquad \theta \neq 90^\circ$$



i. Through two points, $m = \frac{y_2 - y_1}{x_2 - x_1}$.

ii. For $ax + by + c = 0$ with $b \neq 0$, the slope is $-\frac{a}{b}$.

iii. A line equally inclined to both axes has $m = \pm 1$.

iv. Two lines are perpendicular exactly when $m_1 m_2 = -1$; they are parallel when $m_1 = m_2$.

The vertical case. A line parallel to the Y-axis has $\theta = 90^\circ$ and no slope at all. Its equation is $x = k$, and no slope formula applies to it.

13 LINES THROUGH A GIVEN POINT
C·13

A line is fixed by two points, or by one point together with its direction. The one-point forms are:

SLOPE FORM

Through the origin with slope m:

$$y = mx$$

POINT AND SLOPE FORM

Through (x_1, y_1) with slope m:

KEY

$$y - y_1 = m(x - x_1)$$

SLOPE INTERCEPT FORM

Slope m, cutting an intercept c on the positive Y-axis:

$$y = mx + c$$

If instead the line cuts an intercept d on the X-axis, then $y = m(x - d)$.

14 TWO POINT FORM
C·14

Through (x_1, y_1) and (x_2, y_2):

KEY

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Equivalently, in determinant form,

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Two jobs at once. The determinant form is also the collinearity test: put a third point in the first row and it vanishes exactly when the three points lie on one line.

15 INTERCEPT FORM
C·15

If a line cuts intercepts a on the X-axis and b on the Y-axis, then

KEY

$$\frac{x}{a} + \frac{y}{b} = 1$$

The triangle it cuts off with the axes has area $\frac{1}{2}|ab|$.

Reading it off the general form. From $ax + by + c = 0$ the intercepts are $-\frac{c}{a}$ and $-\frac{c}{b}$; the form breaks down for a line through the origin, where $c = 0$.

16 NORMAL FORM
C·16

Let p be the length of the perpendicular from the origin to the line, and let that perpendicular make an angle α with the positive X-axis. Then

KEY

$$x\cos\alpha + y\sin\alpha = p, \qquad p > 0$$

Only $p > 0$. The right-hand side is a length, so it is always taken positive; α is then free to run over the full turn $0 \leq \alpha < 2\pi$ and fixes which side of the origin the line lies on.

17 PARAMETRIC OR DISTANCE FORM
C·17

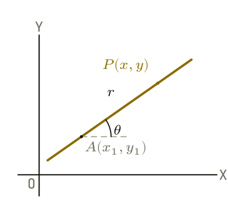
For the line through $A(x_1, y_1)$ making an angle θ with the positive X-axis,

KEY

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = \pm r$$

where r is the distance of (x, y) from (x_1, y_1). It is also called the symmetric or distance form. Any point of the line may therefore be written

$$(x_1 \pm r\cos\theta,\ y_1 \pm r\sin\theta)$$



Why it is worth knowing. It turns a question about a length along a line into a question about one number r, which is what makes chord-length problems on conics tractable.

18 GENERAL EQUATION AND ITS TRANSFORMATIONS
C·18

Every straight line has an equation of the first degree, $ax + by + c = 0$. It is turned into the other standard forms as follows.

TO SLOPE INTERCEPT FORM

KEY

$$ax + by + c = 0 \implies y = -\frac{a}{b}x - \frac{c}{b}$$

so the slope is $-\frac{a}{b}$ and the y-intercept is $-\frac{c}{b}$.

TO INTERCEPT FORM

$$\frac{x}{-\frac{c}{a}} + \frac{y}{-\frac{c}{b}} = 1$$

TO NORMAL FORM

Move c to the right and make it positive, then divide throughout by $\sqrt{a^2 + b^2}$:

$$\frac{-ax}{\sqrt{a^2 + b^2}} + \frac{-by}{\sqrt{a^2 + b^2}} = \frac{c}{\sqrt{a^2 + b^2}}$$

Comparing with $x\cos\alpha + y\sin\alpha = p$,

$$\cos\alpha = \frac{-a}{\sqrt{a^2 + b^2}}, \qquad \sin\alpha = \frac{-b}{\sqrt{a^2 + b^2}}, \qquad p = \frac{c}{\sqrt{a^2 + b^2}}$$

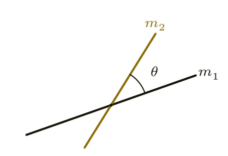
D ANGLE, DISTANCE & POSITION 19-24

19 ANGLE BETWEEN TWO LINES
D·19

If two lines have slopes m_1 and m_2, the angle θ between them satisfies

KEY

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$



For $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ the same thing reads

$$\tan\theta = \left|\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2}\right|$$

- Parallel: $a_1 b_2 - a_2 b_1 = 0$, i.e. $\frac{a_1}{a_2} = \frac{b_1}{b_2}$.
- Perpendicular: $a_1 a_2 + b_1 b_2 = 0$.
- Coincident: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Dropping the modulus. Without the modulus the formula returns the angle measured from the second line to the first, and the two lines make angles θ and $180^\circ - \theta$ with each other. Keep the modulus unless the question asks for a directed angle.

20 A LINE AT A GIVEN ANGLE TO ANOTHER
D·20

The line through (x_1, y_1) making an angle α with the given line $y = mx + c$ is

KEY

$$y - y_1 = \left(\frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}\right)(x - x_1)$$

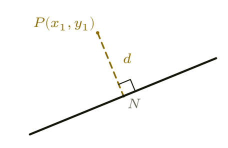
Two answers, not one. The upper and lower signs give the two lines through the point that make the angle α with the given line, one on each side of it. Both are valid unless the problem rules one out.

21 LENGTH OF THE PERPENDICULAR
D·21

The distance of $P(x_1, y_1)$ from the line $ax + by + c = 0$ is

KEY

$$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$



DISTANCE FROM THE ORIGIN

$$d = \frac{|c|}{\sqrt{a^2 + b^2}}$$

DISTANCE BETWEEN TWO PARALLEL LINES

For $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$,

$$d = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

Make them match first. The parallel-line formula only works once the coefficients of x and of y are identical in the two equations. Multiply one equation through before subtracting.

22 POSITION OF A POINT
D·22

Let $L(x, y) = ax + by + c$. If $L(x_1, y_1)$ has the same sign as c, the point (x_1, y_1) lies on the origin side of the line $ax + by + c = 0$; if the signs are opposite, it lies on the non-origin side.

TWO POINTS TOGETHER

$P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the same side of $ax + by + c = 0$ when

KEY

$$\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} > 0$$

and on opposite sides when the same quotient is negative.

Same idea, cleaner. All of this is one statement: L keeps a fixed sign on each side of the line, so comparing signs of L at two points tells you whether the segment joining them crosses the line.

23 FOOT OF THE PERPENDICULAR AND IMAGE
D·23

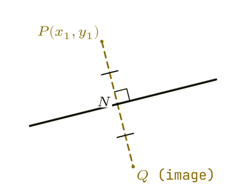
For the point $P(x_1, y_1)$ and the line $ax + by + c = 0$, the foot of the perpendicular $N(h, k)$ is given by

KEY

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = -\frac{ax_1 + by_1 + c}{a^2 + b^2}$$

The image, or reflection, Q of P in the line is obtained by doubling the same step:

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = -\frac{2(ax_1 + by_1 + c)}{a^2 + b^2}$$



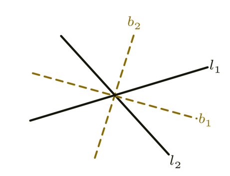
Reflection in the axes and in $y = x$. The image of (x_1, y_1) is $(x_1, -y_1)$ in the X-axis, $(-x_1, y_1)$ in the Y-axis, and (y_1, x_1) in the line $y = x$.

24 BISECTORS OF THE ANGLES
D·24

The bisectors of the angles between $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ are

KEY

$$\frac{a_1 x + b_1 y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2 x + b_2 y + c_2}{\sqrt{a_2^2 + b_2^2}}$$



TELLING THE TWO APART

First make c_1 and c_2 both positive. Then:

i. The $+$ sign gives the bisector of the angle that contains the origin; the $-$ sign gives the other one.

ii. If $a_1 a_2 + b_1 b_2 > 0$, the origin lies in the obtuse angle, so the $+$ bisector is the obtuse one and the $-$ bisector is the acute one. If $a_1 a_2 + b_1 b_2 < 0$, the roles are exchanged.

Why it works. Both sides of the bisector equation are signed distances to the two lines. Equal distances is exactly what a bisector means; the sign chooses which of the two perpendicular bisecting lines you get.

E FAMILIES & CHANGE OF AXES 25-28

25 FAMILY OF LINES
E·25

If $l_1 : a_1 x + b_1 y + c_1 = 0$ and $l_2 : a_2 x + b_2 y + c_2 = 0$ intersect, then every line through their point of intersection can be written as

KEY

$$l_1 + \lambda l_2 = 0$$

that is,

$$(a_1 x + b_1 y + c_1) + \lambda(a_2 x + b_2 y + c_2) = 0$$

One further condition in the problem, such as passing through a given point or having a given slope, fixes λ and picks out the required line.

The one line it misses. No value of λ returns l_2 itself, so check that case separately when the family is used to prove something about all lines through the intersection.

26 CONCURRENCY OF THREE LINES
E·26

The three lines $a_r x + b_r y + c_r = 0$, $r = 1, 2, 3$, are concurrent if and only if

KEY

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Equivalently, constants $\lambda_1, \lambda_2, \lambda_3$, not all zero, exist with

$$\lambda_1 l_1 + \lambda_2 l_2 + \lambda_3 l_3 = 0$$

A vanishing determinant is not enough on its own. It also holds when the three lines are parallel or two of them coincide, so confirm that at least two of the lines actually meet before calling them concurrent.

27 CHANGE OF ORIGIN
E·27

The coordinates of a point, and the equation of a curve, both change when the origin is moved or the axes are turned. Either move is called a transformation of axes.

Let the origin be shifted to (h, k) with no rotation. If (x, y) are the old coordinates of a point and (X, Y) the new ones, then

KEY

$$x = X + h, \qquad y = Y + k$$

i. A point (x, y) has new coordinates $(x - h,\ y - k)$.

ii. The new equation of a curve is found by writing $X + h$ in place of x and $Y + k$ in place of y.

What it is for. Choosing (h, k) well kills the first-degree terms of a second-degree curve, which is the standard first step in reducing a conic to its simplest form.

28 ROTATION OF AXES
E·28

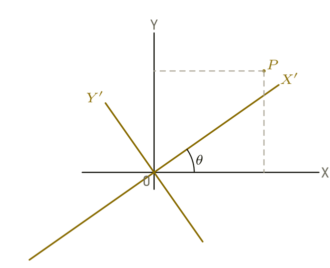
When the axes are turned through an angle θ about the same origin,

KEY

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

and, reading the same relation backwards,

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$



SHIFT AND ROTATION TOGETHER

If the origin first moves to $O'(h, k)$ and the axes are then turned anticlockwise through θ about O', the point $P(x, y)$ has new coordinates (x', y') where

$$x = h + x'\cos\theta - y'\sin\theta$$
$$y = k + x'\sin\theta + y'\cos\theta$$

and, going back to the original axes,

$$x' = (x - h)\cos\theta + (y - k)\sin\theta$$
$$y' = -(x - h)\sin\theta + (y - k)\cos\theta$$