CONCEPT MAP

VECTORS

5 TOPICS · 27 CORE IDEAS · ONE SHEET

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A VECTORS & THEIR ALGEBRA 01-06

01 TYPES OF VECTORS A·01

A scalar has magnitude only; a vector has magnitude and direction and obeys the triangle law of addition. The vector from A to B is written $\overrightarrow{AB}$, its length $|\overrightarrow{AB}|$.

i. **Zero vector 0**: magnitude zero, direction undefined.
ii. **Unit vector $\hat{\mathbf{a}}$**: magnitude one. For $\mathbf{a} \neq \mathbf{0}$, $\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$.
iii. **Equal vectors**: same magnitude and same direction, wherever they are drawn.
iv. **Negative** of **a**: same magnitude, opposite direction, written $-\mathbf{a}$.
v. **Collinear or parallel vectors**: along the same line or parallel lines; $\mathbf{b} = \lambda\mathbf{a}$ for some scalar λ.
vi. **Coplanar vectors**: all parallel to one plane. Any two vectors are always coplanar.
vii. **Co-initial vectors**: drawn from the same initial point.
viii. **Free and localised vectors**: a free vector may be moved anywhere parallel to itself; a localised one is tied to a fixed line of action.

> **Why the zero vector has no direction.** Every direction serves it equally, so no single one can be assigned. This is the reason the angle between **0** and any vector is left undefined, and both $\mathbf{0} \cdot \mathbf{a}$ and $\mathbf{0} \times \mathbf{a}$ are defined separately rather than derived.

02 ADDITION OF VECTORS A·02

TRIANGLE LAW

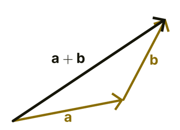
Place the tail of **b** at the head of **a**; the vector closing the triangle from the tail of **a** to the head of **b** is $\mathbf{a} + \mathbf{b}$.

KEY

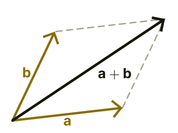
$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

PARALLELOGRAM LAW

If **a** and **b** are two adjacent sides of a parallelogram drawn from the same point, their sum is the diagonal through that point.



triangle law



parallelogram law

PROPERTIES

i. Commutative: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.
ii. Associative: $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$.
iii. Identity: $\mathbf{a} + \mathbf{0} = \mathbf{a}$; inverse: $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$.
iv. Polygon law: vectors placed head to tail round a closed polygon sum to **0**.

> **Two inequalities worth memorising.** $||\mathbf{a}| - |\mathbf{b}|| \leq |\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$. Equality on the right needs **a** and **b** in the same direction.

03 MULTIPLICATION BY A SCALAR A·03

For a scalar m and a vector **a**, the product $m\mathbf{a}$ has magnitude $|m||\mathbf{a}|$, direction that of **a** when $m > 0$ and the opposite when $m < 0$.

KEY

$$|m\mathbf{a}| = |m|\,|\mathbf{a}|$$

i. $m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}$ and $(m + n)\mathbf{a} = m\mathbf{a} + n\mathbf{a}$.
ii. $m(n\mathbf{a}) = (mn)\mathbf{a}$.
iii. **a** and **b** are parallel exactly when $\mathbf{b} = m\mathbf{a}$ for some scalar m.

> **Building a vector to order.** A vector of magnitude k in the direction of **a** is $k\hat{\mathbf{a}} = \frac{k\,\mathbf{a}}{|\mathbf{a}|}$. Almost every "find the vector of length k along ..." question is this one line.

04 POSITION VECTOR AND SECTION FORMULA A·04

Fix an origin O. The position vector of a point P is $\overrightarrow{OP}$. For points A and B with position vectors **a** and **b**,

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

If R divides AB in the ratio $m : n$ internally, its position vector is

KEY

$$\mathbf{r} = \frac{m\mathbf{b} + n\mathbf{a}}{m + n}$$

For external division replace n by $-n$, giving $\frac{m\mathbf{b} - n\mathbf{a}}{m - n}$. The mid-point is $\frac{\mathbf{a} + \mathbf{b}}{2}$.

> **Centroid.** The centroid of the triangle with vertices **a**, **b**, **c** is $\frac{\mathbf{a} + \mathbf{b} + \mathbf{c}}{3}$, and of the tetrahedron with vertices **a**, **b**, **c**, **d** it is $\frac{\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}}{4}$.

05 COMPONENTS AND DIRECTION COSINES A·05

With $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ the unit vectors along the coordinate axes, every vector is written

KEY

$$\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}, \qquad |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

If **a** makes angles α, β, γ with the axes, its direction cosines are

$$l = \cos\alpha = \frac{a_1}{|\mathbf{a}|}, \quad m = \cos\beta = \frac{a_2}{|\mathbf{a}|}, \quad n = \cos\gamma = \frac{a_3}{|\mathbf{a}|}$$

$$l^2 + m^2 + n^2 = 1$$

Any three numbers proportional to l, m, n are direction ratios of **a**; the components a_1, a_2, a_3 themselves are one such set.

06 COLLINEARITY, COPLANARITY AND INDEPENDENCE A·06

A vector $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$ is called a linear combination of **a**, **b**, **c**. Vectors are linearly independent when the only combination equal to **0** is the one with all coefficients zero.

KEY

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0} \implies x = y = z = 0$$

i. Two vectors are collinear iff one is a scalar multiple of the other; then they are dependent.
ii. Three points **a**, **b**, **c** are collinear iff $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0}$ with $x + y + z = 0$, not all zero.
iii. Three vectors are coplanar iff one is a linear combination of the other two.
iv. Any three non-coplanar vectors form a basis: every vector in space is uniquely $x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$.

> **Why the basis matters.** Uniqueness is what lets you compare coefficients on both sides of a vector equation, which is how most "find x, y, z" problems are actually solved.

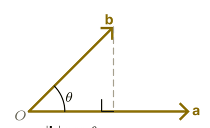
B THE SCALAR PRODUCT 07-13

07 THE SCALAR OR DOT PRODUCT B·07

Let **a** and **b** be two non-zero vectors inclined at an angle θ. Their scalar product is the number

KEY

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$$



> **The zero vector.** If **a** or **b** is **0** then θ is undefined, because **0** has no direction. In that case the dot product is defined outright to be the scalar zero.

08 PROJECTION B·08

The scalar product of two vectors equals the modulus of either one times the projection of the other in its direction. Hence

KEY

$$\text{projection of } \mathbf{b} \text{ on } \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \hat{\mathbf{a}} \cdot \mathbf{b}$$

$$\text{projection of } \mathbf{a} \text{ on } \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \mathbf{a} \cdot \hat{\mathbf{b}}$$

> **It is a signed length.** The projection is negative when θ is obtuse, so it is not a distance. Take the modulus only if the question asks for the length of the projection.

09 PROPERTIES OF THE SCALAR PRODUCT B·09

i. **Commutative**: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.
ii. **Distributive** over addition: $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ and $(\mathbf{b} + \mathbf{c}) \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{a}$.
iii. For non-zero **a**, **b**: $\mathbf{a} \cdot \mathbf{b} = 0 \iff \mathbf{a} \perp \mathbf{b}$.
iv. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$.
v. $(m\mathbf{a}) \cdot \mathbf{b} = m(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (m\mathbf{b})$ for a scalar m.
vi. $m\mathbf{a} \cdot n\mathbf{b} = mn(\mathbf{a} \cdot \mathbf{b})$ for scalars m, n.
vii. $\mathbf{a} \cdot (-\mathbf{b}) = -(\mathbf{a} \cdot \mathbf{b})$ and $(-\mathbf{a}) \cdot (-\mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$.

THE ORTHOGONAL TRIAD

KEY

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$$

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$$

> **Correction to the printed page.** The source prints left distributivity as $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$. The second sign must be a plus, as written above.

10 THE NORM IDENTITIES B·10

Expanding $|\mathbf{a} \pm \mathbf{b}|^2 = (\mathbf{a} \pm \mathbf{b}) \cdot (\mathbf{a} \pm \mathbf{b})$ gives the whole family:

KEY

$$|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\,\mathbf{a} \cdot \mathbf{b}$$

$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,\mathbf{a} \cdot \mathbf{b}$$

i. $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2$.
ii. $|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2)$ [parallelogram law].
iii. $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}| \iff \mathbf{a} \perp \mathbf{b}$.
iv. $|\mathbf{a} + \mathbf{b}| = |\mathbf{a}| + |\mathbf{b}| \iff$ **a** is parallel to **b**.
v. $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 \iff$ **a** and **b** are orthogonal.

THREE VECTORS

$$|\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a})$$

11 SCALAR PRODUCT IN COMPONENTS B·11

Let $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$. Expanding by distributivity and using the triad relations,

KEY

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

> **Everything follows from this.** Perpendicularity becomes $a_1b_1 + a_2b_2 + a_3b_3 = 0$, and the modulus becomes $\sqrt{a_1^2 + a_2^2 + a_3^2}$, so a component question never needs the angle.

12 ANGLE BETWEEN TWO VECTORS B·12

From $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$,

KEY

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}, \qquad \theta = \cos^{-1}\left[\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}\right]$$

In components this reads

$$\theta = \cos^{-1}\left[\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}\right]$$

13 COMPONENTS ALONG AND PERPENDICULAR TO A VECTOR B·13

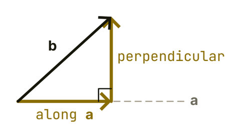
Any vector **b** splits uniquely into a part parallel to **a** and a part perpendicular to it. The component of **b** along **a** is

KEY

$$\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$$

and the component perpendicular to **a** is whatever is left:

$$\mathbf{b} - \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$$



> **Do not confuse the two.** The projection in section 08 is a number; the component here is a vector. They differ by the factor $\hat{\mathbf{a}}$.

C THE VECTOR PRODUCT 14-18

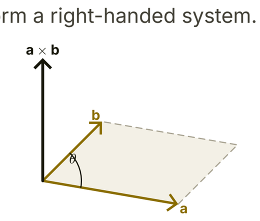
14 THE VECTOR OR CROSS PRODUCT C·14

Let **a** and **b** be non-zero, non-parallel vectors inclined at θ. Their vector product $\mathbf{a} \times \mathbf{b}$ is the vector

KEY

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin\theta\,\hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is the unit vector perpendicular to the plane of **a** and **b**, chosen so that **a**, **b**, $\hat{\mathbf{n}}$ form a right-handed system.



- If **a** or **b** is **0** then θ, and so $\hat{\mathbf{n}}$, is undefined; the product is defined to be **0**.
- If **a** and **b** are collinear, $\theta = 0$ or π, the direction of $\hat{\mathbf{n}}$ is not determined and again $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

15 PROPERTIES OF THE VECTOR PRODUCT C·15

i. **Not commutative**: $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$; instead $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$.
ii. $m(\mathbf{a} \times \mathbf{b}) = (m\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (m\mathbf{b})$ for a scalar m.
iii. $m\mathbf{a} \times n\mathbf{b} = mn(\mathbf{a} \times \mathbf{b})$ for scalars m, n.
iv. **Distributive**: $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ and $(\mathbf{b} + \mathbf{c}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}$.
v. $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c}$.
vi. For non-zero vectors, $\mathbf{a} \times \mathbf{b} = \mathbf{0} \iff \mathbf{a} \parallel \mathbf{b}$. In particular $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.

> **Order is everything.** Because the product anticommutes, the sign of every cross-product answer depends on the order you wrote the factors in. Fix the order at the start of a problem and keep it.

16 THE ORTHOGONAL TRIAD C·16

For the unit vectors along the axes,

KEY

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \mathbf{0}$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}, \quad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}, \quad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}, \quad \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}, \quad \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$$

> **The cycle.** Read $\hat{\mathbf{i}} \to \hat{\mathbf{j}} \to \hat{\mathbf{k}} \to \hat{\mathbf{i}}$ round a circle. Going forward gives a plus sign, going backward a minus.

17 CROSS PRODUCT IN COMPONENTS C·17

For $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$,

KEY

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

UNIT NORMAL TO A PLANE

The unit vectors normal to the plane containing two non-zero, non-parallel vectors **a** and **b** are

$$\pm\frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$$

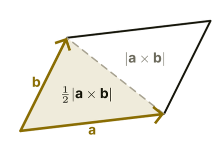
> **Correction to the printed page.** The source writes this as $\pm\frac{\mathbf{a} \times \mathbf{b}}{\mathbf{a} \times \mathbf{b}}$. The denominator must be the modulus $|\mathbf{a} \times \mathbf{b}|$, a scalar; a vector cannot divide a vector.

18 AREAS FROM THE CROSS PRODUCT C·18

Since $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$ is exactly base times height, every plane area on this sheet is a cross product.

KEY

$$\text{parallelogram with sides } \mathbf{a}, \mathbf{b} = |\mathbf{a} \times \mathbf{b}|$$



i. Triangle with sides **a** and **b**: $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.
ii. Triangle ABC: $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}|\overrightarrow{BC} \times \overrightarrow{BA}| = \frac{1}{2}|\overrightarrow{CB} \times \overrightarrow{CA}|$.
iii. Parallelogram with diagonals **a** and **b**: $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.
iv. Plane quadrilateral $ABCD$: $\frac{1}{2}|\overrightarrow{AC} \times \overrightarrow{BD}|$, where AC and BD are its diagonals.

D TRIPLE PRODUCTS 19-25

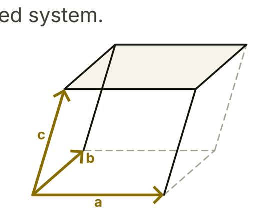
19 SCALAR TRIPLE PRODUCT D·19

For three vectors **a**, **b**, **c** the number $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is called their scalar triple product and is written $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

KEY

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

It is the volume of the parallelepiped whose coterminous edges **a**, **b**, **c** form a right-handed system.



> **Sign and volume.** The triple product is a signed number: negative when the three edges form a left-handed system. A volume is $|[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]|$.

20 PROPERTIES OF THE SCALAR TRIPLE PRODUCT D·20

i. **Cyclic permutation leaves it unchanged**: $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$, that is $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{b}\ \mathbf{c}\ \mathbf{a}] = [\mathbf{c}\ \mathbf{a}\ \mathbf{b}]$.
ii. **Breaking the cycle reverses the sign**, not the magnitude: $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = -[\mathbf{b}\ \mathbf{a}\ \mathbf{c}] = -[\mathbf{c}\ \mathbf{b}\ \mathbf{a}] = -[\mathbf{a}\ \mathbf{c}\ \mathbf{b}]$.
iii. **Dot and cross may be interchanged** provided the cyclic order stands: $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.
iv. It vanishes if any two of the vectors are equal: $[\mathbf{a}\ \mathbf{a}\ \mathbf{c}] = 0$.
v. For scalars λ, μ, ν: $[\lambda\mathbf{a}\ \mu\mathbf{b}\ \nu\mathbf{c}] = \lambda\mu\nu[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.
vi. It vanishes if any two of the vectors are parallel or collinear.
vii. It is additive in each slot: $[\mathbf{a} + \mathbf{b}\ \mathbf{c}\ \mathbf{d}] = [\mathbf{a}\ \mathbf{c}\ \mathbf{d}] + [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]$.

21 COMPONENTS AND THE COPLANARITY TEST D·21

With $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, and **b**, **c** written the same way,

KEY

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

COPLANARITY

Three non-zero, non-collinear vectors are coplanar if and only if the parallelepiped they span is flat:

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$$

Four points with position vectors **a**, **b**, **c**, **d** are coplanar if

$$[\mathbf{d}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{d}\ \mathbf{c}\ \mathbf{a}] + [\mathbf{d}\ \mathbf{a}\ \mathbf{b}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

22 FURTHER TRIPLE PRODUCT IDENTITIES D·22

If **a**, **b**, **c** are non-coplanar and $\mathbf{A} = x_1\mathbf{a} + y_1\mathbf{b} + z_1\mathbf{c}$, $\mathbf{B} = x_2\mathbf{a} + y_2\mathbf{b} + z_2\mathbf{c}$, $\mathbf{C} = x_3\mathbf{a} + y_3\mathbf{b} + z_3\mathbf{c}$, then

KEY

$$[\mathbf{A}\ \mathbf{B}\ \mathbf{C}] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

For two triads **a**, **b**, **c** and **l**, **m**, **n**,

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}][\mathbf{l}\ \mathbf{m}\ \mathbf{n}] = \begin{vmatrix} \mathbf{a} \cdot \mathbf{l} & \mathbf{b} \cdot \mathbf{l} & \mathbf{c} \cdot \mathbf{l} \\ \mathbf{a} \cdot \mathbf{m} & \mathbf{b} \cdot \mathbf{m} & \mathbf{c} \cdot \mathbf{m} \\ \mathbf{a} \cdot \mathbf{n} & \mathbf{b} \cdot \mathbf{n} & \mathbf{c} \cdot \mathbf{n} \end{vmatrix}$$

Taking the two triads to be the same gives the Gram determinant:

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2 = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{c} \cdot \mathbf{a} & \mathbf{c} \cdot \mathbf{b} & \mathbf{c} \cdot \mathbf{c} \end{vmatrix}$$

and, for vectors **a**, **b**, **c**, **l**, **m**,

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}](\mathbf{l} \times \mathbf{m}) = \begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a} \cdot \mathbf{l} & \mathbf{b} \cdot \mathbf{l} & \mathbf{c} \cdot \mathbf{l} \\ \mathbf{a} \cdot \mathbf{m} & \mathbf{b} \cdot \mathbf{m} & \mathbf{c} \cdot \mathbf{m} \end{vmatrix}$$

23 VECTOR TRIPLE PRODUCT D·23

For any three vectors, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ are called the vector triple products of **a**, **b**, **c**. The expansion is

KEY

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$$

AID TO MEMORY

Number the vectors in the order they appear and the rule reads

$$\mathrm{I} \times (\mathrm{II} \times \mathrm{III}) = (\mathrm{I} \cdot \mathrm{III})\,\mathrm{II} - (\mathrm{I} \cdot \mathrm{II})\,\mathrm{III}$$

> **Middle vector first.** The surviving positive term always carries the vector in the middle of the bracket. That single cue fixes the whole formula.

24 PROPERTIES OF THE VECTOR TRIPLE PRODUCT D·24

i. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is a linear combination of the two vectors inside the bracket.
ii. The vector $\mathbf{r} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is perpendicular to **a** and lies in the plane of **b** and **c**.
iii. The expansion holds only when the vector outside the bracket is written on the left. If it is not, move it there first using anti-commutativity.
iv. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$, so these three vectors are coplanar.

MOVING THE BRACKET

KEY

$$\begin{aligned}(\mathbf{b} \times \mathbf{c}) \times \mathbf{a} &= -\{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})\} \\ &= -\{(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}\} \\ &= -(\mathbf{a} \cdot \mathbf{c})\mathbf{b} + (\mathbf{a} \cdot \mathbf{b})\mathbf{c}\end{aligned}$$

> **The product is not associative.** $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ are different vectors in general; the brackets can never be dropped.

25 SOLVING A VECTOR EQUATION D·25

A standard problem gives one dot condition and one cross condition and asks for the unknown vector. Let $\mathbf{a} \neq \mathbf{0}$ and suppose

$$\mathbf{a} \cdot \mathbf{r} = p, \qquad \mathbf{a} \times \mathbf{r} = \mathbf{q}$$

Crossing the second equation with **a** and expanding the vector triple product gives $\mathbf{a} \times (\mathbf{a} \times \mathbf{r}) = (\mathbf{a} \cdot \mathbf{r})\mathbf{a} - |\mathbf{a}|^2\mathbf{r}$, so

KEY

$$\mathbf{r} = \frac{p\,\mathbf{a} - \mathbf{a} \times \mathbf{q}}{|\mathbf{a}|^2}$$

> **Consistency.** A solution exists only when $\mathbf{a} \cdot \mathbf{q} = 0$, since $\mathbf{q} = \mathbf{a} \times \mathbf{r}$ is perpendicular to **a** by construction. Check that before solving.

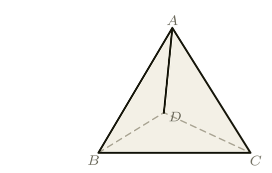
E VOLUME & RECIPROCAL SYSTEM 26-27

26 VOLUME OF A TETRAHEDRON E·26

The volume of the tetrahedron whose three coterminous edges are **a**, **b**, **c** in the right-handed system is one sixth of the parallelepiped on the same edges:

KEY

$$V = \tfrac{1}{6}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$



If A, B, C, D are the vertices of a tetrahedron $ABCD$, its volume is numerically

$$\tfrac{1}{6}[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = \tfrac{1}{6}[\overrightarrow{BC}\ \overrightarrow{BD}\ \overrightarrow{BA}] = \tfrac{1}{6}[\overrightarrow{CD}\ \overrightarrow{CB}\ \overrightarrow{CA}]$$

> **Flat means coplanar.** The four vertices are coplanar exactly when this volume is zero, which is the same coplanarity test as in section 21.

27 RECIPROCAL SYSTEM OF VECTORS E·27

Let **a**, **b**, **c** be non-coplanar, so $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0$, and define

KEY

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}, \qquad \mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}, \qquad \mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

Then **a**′, **b**′, **c**′ are said to form the reciprocal system of vectors for **a**, **b**, **c**.

WHAT MAKES IT RECIPROCAL

i. $\mathbf{a} \cdot \mathbf{a}' = \mathbf{b} \cdot \mathbf{b}' = \mathbf{c} \cdot \mathbf{c}' = 1$, and every other dot product between the two systems is zero.
ii. $[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}'] = \frac{1}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$.
iii. $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are their own reciprocal system.