CONCEPT MAP

CIRCLE

5 TOPICS · 30 CORE IDEAS · ONE SHEET

IITIANFORUM
REAL STUDY, GENUINE RESULTS.
Ritesh Raj · M.Sc Mathematics, IIT Delhi
JEE Main · JEE Advanced · Olympiad Mathematics
iitianforum.com

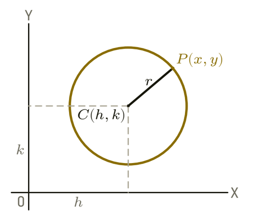
A EQUATION OF A CIRCLE 01-06

01 STANDARD FORM (A·01)

A circle is the locus of a point that moves at a constant distance r, the radius, from a fixed point $C(h, k)$, the centre. Squaring the distance formula gives the standard, or centre-radius, form:

KEY

$$(x-h)^2+(y-k)^2=r^2$$



i. Centre at the origin: $x^2+y^2=r^2$.
ii. Centre on the X-axis: $(x-h)^2+y^2=r^2$.
iii. Passing through the origin with centre (h,k): $x^2+y^2-2hx-2ky=0$.

02 GENERAL EQUATION (A·02)

Expanding the standard form gives the general second-degree equation

KEY

$$S \equiv x^2+y^2+2gx+2fy+c=0$$

whose centre and radius are

$$C=(-g,-f), \qquad r=\sqrt{g^2+f^2-c}$$

THREE CASES

i. $g^2+f^2-c>0$: a real circle.
ii. $g^2+f^2-c=0$: a point circle, the single point $(-g,-f)$.
iii. $g^2+f^2-c<0$: an imaginary circle, no real locus.

How to recognise a circle. A second-degree equation $ax^2+by^2+2hxy+2gx+2fy+c=0$ is a circle only when $a=b\neq 0$ and $h=0$. Divide through by a before reading off the centre.

03 DIAMETER FORM (A·03)

If $A(x_1,y_1)$ and $B(x_2,y_2)$ are the ends of a diameter, then for any point $P(x,y)$ on the circle the angle APB is a right angle, so $PA \perp PB$. Writing that with slopes gives

KEY

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

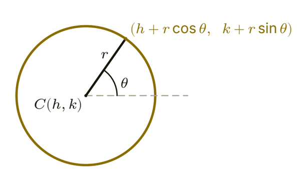
Angle in a semicircle. The whole derivation is the classical theorem that the angle subtended by a diameter at any point of the circle is 90°.

04 PARAMETRIC FORM (A·04)

Every point of the circle $(x-h)^2+(y-k)^2=r^2$ can be written with one parameter θ, the angle the radius makes with the positive X-direction:

KEY

$$x=h+r\cos\theta, \qquad y=k+r\sin\theta, \qquad 0\le\theta<2\pi$$



For $x^2+y^2=r^2$ this is simply $(r\cos\theta,\ r\sin\theta)$, often written as the point θ.

Chord joining two parameters. The chord through the points α and β on $x^2+y^2=r^2$ is $x\cos\frac{\alpha+\beta}{2}+y\sin\frac{\alpha+\beta}{2}=r\cos\frac{\alpha-\beta}{2}$.

05 CIRCLE THROUGH THREE POINTS (A·05)

Three non-collinear points determine exactly one circle. Substitute each point into $x^2+y^2+2gx+2fy+c=0$ and solve the three linear equations for g, f, c.

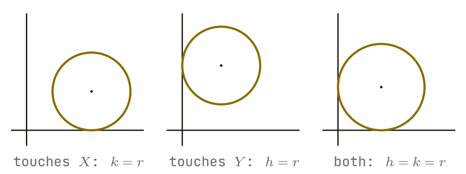
The same circle is written directly as the determinant

KEY

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix}=0$$

Faster by geometry. The centre is the intersection of the perpendicular bisectors of any two of the three chords; the radius is then its distance to any one vertex.

06 CIRCLES IN SPECIAL POSITIONS (A·06)



i. Touching the X-axis: $|k|=r$, so $(x-h)^2+(y-k)^2=k^2$.
ii. Touching the Y-axis: $|h|=r$, so $(x-h)^2+(y-k)^2=h^2$.
iii. Touching both axes: $|h|=|k|=r$, giving $(x\pm r)^2+(y\pm r)^2=r^2$.
iv. Passing through the origin: $c=0$.
v. Centre on the X-axis: $f=0$; centre on the Y-axis: $g=0$.

INTERCEPTS ON THE AXES

The circle $x^2+y^2+2gx+2fy+c=0$ cuts chords on the axes of lengths

KEY

$$\text{on } X:\ 2\sqrt{g^2-c}, \qquad \text{on } Y:\ 2\sqrt{f^2-c}$$

Concentric circles. Circles with the same centre differ only in c: $x^2+y^2+2gx+2fy+\lambda=0$ is concentric with $S=0$ for every λ.

B POINT, LINE & CIRCLE 07-13

07 POSITION OF A POINT (B·07)

Write S_1 for the value of $S\equiv x^2+y^2+2gx+2fy+c$ at the point $P(x_1,y_1)$:

KEY

$$S_1=x_1^2+y_1^2+2gx_1+2fy_1+c$$

i. $S_1>0$: P lies outside the circle.
ii. $S_1=0$: P lies on the circle.
iii. $S_1<0$: P lies inside the circle.

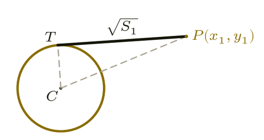
Power of a point. S_1 is exactly CP^2-r^2, the power of P with respect to the circle. Every result in this topic is a restatement of that one quantity.

08 LENGTH OF THE TANGENT (B·08)

From an external point $P(x_1,y_1)$ the two tangents to $S=0$ have equal length

KEY

$$PT=\sqrt{S_1}=\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$



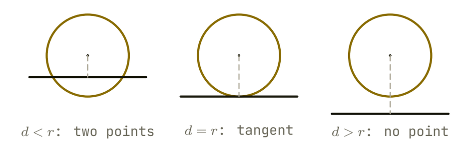
Coefficients must be unity. The formula holds only when the coefficients of x^2 and y^2 are both 1. Divide the equation through first, or the length comes out wrong by a factor.

09 A LINE AND A CIRCLE (B·09)

Let d be the distance of the centre from the line $y=mx+c_1$, or $lx+my+n=0$, and r the radius:

KEY

$$d=\frac{|lh+mk+n|}{\sqrt{l^2+m^2}}$$



i. $d<r$: the line is a secant and cuts the circle in two points.
ii. $d=r$: the line touches the circle; it is a tangent.
iii. $d>r$: the line misses the circle entirely.

Condition of tangency. $y=mx+c_1$ touches $x^2+y^2=r^2$ exactly when $c_1^2=r^2(1+m^2)$, that is $c_1=\pm r\sqrt{1+m^2}$.

10 LENGTH OF A CHORD (B·10)

A chord at distance d from the centre of a circle of radius r has length

KEY

$$2\sqrt{r^2-d^2}$$

- The longest chord is the diameter, of length $2r$, at $d=0$.
- If a chord subtends an angle 2α at the centre, its length is $2r\sin\alpha$.
- The perpendicular from the centre bisects the chord.

Chord of a given length. A chord of length ℓ lies at distance $d=\sqrt{r^2-\ell^2/4}$ from the centre, so the family of such chords touches the concentric circle of radius d.

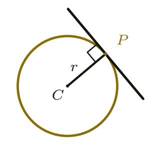
11 EQUATION OF THE TANGENT (B·11)

POINT FORM

The tangent at a point (x_1,y_1) of $S=0$ is $T=0$, that is

KEY

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$$



the tangent is perpendicular to the radius

SLOPE FORM

The tangents of slope m to $x^2+y^2=r^2$ are

$$y=mx\pm r\sqrt{1+m^2}$$

touching at the points $\left(\mp\frac{rm}{\sqrt{1+m^2}},\ \pm\frac{r}{\sqrt{1+m^2}}\right)$.

PARAMETRIC FORM

The tangent at the point θ of $x^2+y^2=r^2$ is

$$x\cos\theta+y\sin\theta=r$$

12 EQUATION OF THE NORMAL (B·12)

The normal at a point of a circle is the line through that point and the centre, since the tangent is perpendicular to the radius. At (x_1,y_1) on $S=0$ the normal is

KEY

$$\frac{x-x_1}{x_1+g}=\frac{y-y_1}{y_1+f}$$

For $x^2+y^2=r^2$ this reduces to $y=\frac{y_1}{x_1}x$, a line through the origin.

Every normal passes through the centre. That single fact answers most normal questions on circles without any algebra.

13 PAIR OF TANGENTS AND CHORD OF CONTACT (B·13)

PAIR OF TANGENTS

The two tangents drawn from an external point $P(x_1,y_1)$ together form the pair

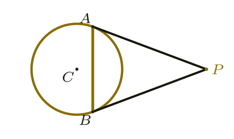
KEY

$$SS_1=T^2$$

CHORD OF CONTACT

The chord joining the two points of contact is

$$T=0$$



chord of contact AB is T = 0

Its length is $\frac{2r\sqrt{S_1}}{\sqrt{S_1+r^2}}$, and the area of the triangle PAB is $\frac{rS_1^{3/2}}{S_1+r^2}$.

Angle between the tangents. If 2α is the angle between them, $\tan\alpha=\frac{r}{\sqrt{S_1}}$.

C CHORDS, POLARS & FAMILIES 14-19

14 CHORD WITH A GIVEN MID-POINT (C·14)

The chord of $S=0$ whose mid-point is (x_1,y_1) is

KEY

$$T=S_1$$

written out, $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=x_1^2+y_1^2+2gx_1+2fy_1+c$.

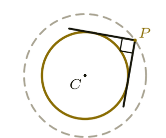
Why it works. The mid-point of a chord and the centre determine the chord's direction, because the line from the centre to the mid-point is perpendicular to the chord.

15 DIRECTOR CIRCLE (C·15)

The locus of a point from which two perpendicular tangents can be drawn to a circle is a concentric circle of radius $r\sqrt{2}$, called the director circle.

KEY

$$x^2+y^2=2r^2 \quad \text{for} \quad x^2+y^2=r^2$$



director circle, radius $r\sqrt{2}$

For the general circle $S=0$ the director circle is $x^2+y^2+2gx+2fy+2c-g^2-f^2=0$.

One line of proof. If the tangents from P are perpendicular, $CTPT'$ is a square of side r, so $CP=r\sqrt{2}$.

16 POLE AND POLAR (C·16)

For any point $P(x_1,y_1)$, inside or outside, the line

KEY

$$T=0$$

is called the polar of P with respect to the circle, and P is the pole of that line.

i. If P is outside, the polar is its chord of contact.
ii. If P is on the circle, the polar is the tangent at P.
iii. If P is inside, the polar lies entirely outside the circle.
iv. The pole of $lx+my+n=0$ with respect to $x^2+y^2=r^2$ is $\left(-\frac{lr^2}{n}, -\frac{mr^2}{n}\right)$.

Conjugate points. If the polar of P passes through Q, then the polar of Q passes through P; such points are called conjugate.

17 FAMILY OF CIRCLES (C·17)

i. Through the intersection of the circle $S=0$ and the line $L=0$: $S+\lambda L=0$.
ii. Through the intersection of two circles $S_1=0$ and $S_2=0$: $S_1+\lambda S_2=0$, $\lambda\neq-1$.
iii. Touching the line $L=0$ at the point (x_1,y_1) on it: $(x-x_1)^2+(y-y_1)^2+\lambda L=0$.
iv. Touching the circle $S=0$ at (x_1,y_1): $S+\lambda T=0$.
v. Through two given points A and B: $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+\lambda\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$.

The excluded value. In case 2, $\lambda=-1$ makes the second-degree terms cancel and leaves the common chord, a line rather than a circle.

18 COMMON CHORD OF TWO CIRCLES (C·18)

For two intersecting circles $S_1=0$ and $S_2=0$, each written with unit coefficients of x^2 and y^2, subtracting kills the square terms and leaves the common chord:

KEY

$$S_1-S_2=0$$

If d is the distance between the centres and r_1, r_2 the radii, the length of the common chord is

$$2\sqrt{r_1^2-p_1^2} \quad \text{where} \quad p_1=\frac{|r_1^2-r_2^2+d^2|}{2d}$$

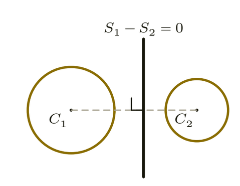
Longest when it is a diameter. The common chord is greatest when it passes through the centre of the smaller circle, in which case its length is that circle's diameter.

19 RADICAL AXIS AND RADICAL CENTRE (C·19)

The radical axis of two circles is the locus of points having equal power with respect to both. Subtracting the two equations gives it directly:

KEY

$$S_1-S_2=0$$



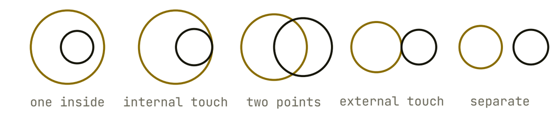
i. It is always perpendicular to the line joining the centres.
ii. If the circles cut, it is their common chord; if they touch, it is the common tangent at the point of contact.
iii. It bisects every common tangent of the two circles.
iv. For three circles taken in pairs the three radical axes are concurrent, at the radical centre.

What the radical centre is for. It is the centre of the unique circle cutting all three given circles orthogonally, and its radius is the tangent length from it to any of them.

D TWO CIRCLES 20-25

20 RELATIVE POSITION OF TWO CIRCLES (D·20)

Let d be the distance between the centres and r_1, r_2 the radii.



CONDITION	POSITION
$d>r_1+r_2$	one outside the other
$d=r_1+r_2$	touch externally
$\lvert r_1-r_2\rvert<d<r_1+r_2$	cut in two points
$d=\lvert r_1-r_2\rvert$	touch internally
$d<\lvert r_1-r_2\rvert$	one lies inside the other

21 NUMBER OF COMMON TANGENTS (D·21)

POSITION	TANGENTS
one outside the other	4
touch externally	3
cut in two points	2
touch internally	1
one inside the other	0

Reading it. Two direct (external) tangents exist whenever neither circle lies inside the other; the two transverse (internal) tangents need the circles to be entirely outside each other.

22 FINDING THE COMMON TANGENTS (D·22)

The direct common tangents meet at the external centre of similitude P_1, which divides C_1C_2 externally in the ratio $r_1:r_2$; the transverse ones meet at the internal centre P_2, dividing it internally in the same ratio.

KEY

$$P_1=\frac{r_1C_2-r_2C_1}{r_1-r_2}, \qquad P_2=\frac{r_1C_2+r_2C_1}{r_1+r_2}$$

Write a line of slope m through the relevant centre of similitude and impose the tangency condition on either circle; the two values of m give the pair.

Equal radii. If $r_1=r_2$ the external centre goes to infinity: the two direct common tangents are parallel to C_1C_2.

23 LENGTH OF THE COMMON TANGENTS (D·23)

DIRECT COMMON TANGENT

KEY

$$L_{\text{direct}}=\sqrt{d^2-(r_1-r_2)^2}$$

TRANSVERSE COMMON TANGENT

$$L_{\text{transverse}}=\sqrt{d^2-(r_1+r_2)^2}$$

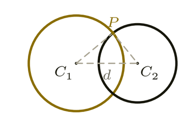
When each is defined. The direct length is real only when $d\ge|r_1-r_2|$, and the transverse length only when $d\ge r_1+r_2$ — which is exactly the tangent count in section 21.

24 ANGLE OF INTERSECTION AND ORTHOGONALITY (D·24)

The angle between two intersecting circles is the angle θ between their tangents at a common point, which is also the angle between the radii there. By the cosine rule in $\triangle C_1PC_2$,

KEY

$$\cos\theta=\frac{r_1^2+r_2^2-d^2}{2r_1r_2}$$



ORTHOGONAL CIRCLES

The circles cut at right angles when $d^2=r_1^2+r_2^2$, which in general form is

$$2g_1g_2+2f_1f_2=c_1+c_2$$

A shortcut for orthogonality. If two circles are orthogonal, the tangent length from either centre to the other circle equals that circle's own radius.

25 CIRCLES TOUCHING A LINE OR A CIRCLE (D·25)

i. A circle of radius r touching the line $lx+my+n=0$ has its centre on either of the two lines parallel to it at distance r.
ii. Circles touching both axes have centre $(\pm a,\pm a)$ and radius $|a|$.
iii. If two circles touch, the point of contact divides C_1C_2 in the ratio $r_1:r_2$ — internally for external contact, externally for internal contact.
iv. The point of contact is $\frac{r_1C_2+r_2C_1}{r_1+r_2}$ for external contact and $\frac{r_1C_2-r_2C_1}{r_1-r_2}$ for internal contact.

Contact means collinear. When two circles touch, the two centres and the point of contact lie on one straight line — which is why the section formula does all the work.

E STANDARD RESULTS & LOCI 26-30

26 CIRCUMCIRCLE AND INCIRCLE OF A TRIANGLE (E·26)

CIRCUMCIRCLE

The circle through the three vertices. Its centre is the intersection of the perpendicular bisectors of the sides, and for a triangle with sides a, b, c and area Δ,

KEY

$$R=\frac{abc}{4\Delta}$$

INCIRCLE

The circle touching all three sides from inside. Its centre is the meeting point of the internal angle bisectors, and

$$r=\frac{\Delta}{s}, \qquad s=\frac{a+b+c}{2}$$

Right-angled triangle. The circumcentre is the mid-point of the hypotenuse, so R is half the hypotenuse; and the circle on the hypotenuse as diameter passes through the right-angled vertex.

27 STANDARD LOCI (E·27)

APOLLONIUS CIRCLE

The locus of a point whose distances from two fixed points A and B are in a constant ratio $k\neq1$ is a circle:

KEY

$$\frac{PA}{PB}=k\ (k\neq1) \implies P \text{ lies on a circle}$$

When $k=1$ the locus degenerates into the perpendicular bisector of AB.

TWO MORE

i. The locus of a point at which AB subtends a right angle is the circle on AB as diameter.
ii. The locus of a point at which AB subtends a constant angle α is a pair of arcs through A and B.

28 CONCYCLIC POINTS (E·28)

Four points are concyclic when one circle passes through all of them. The practical test is to write the circle through three of them and check that the fourth satisfies it.

TWO CURVES MEETING IN FOUR POINTS

The four points where the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ meet the pair of lines $a_3x+b_3y+c_3=0$ and $a_4x+b_4y+c_4=0$ are concyclic if

KEY

$$a_1a_2-b_1b_2=a_3a_4-b_3b_4$$
$$a_1b_2+a_2b_1=a_3b_4+a_4b_3$$

Where the condition comes from. The combined equation of the four lines is a second-degree curve; forcing it to be a circle means equating the x^2 and y^2 coefficients and killing the xy term.

29 DISTANCES, IMAGES AND REFLECTIONS (E·29)

For a point P at distance D from the centre of a circle of radius r:

KEY

$$\text{greatest}=D+r, \qquad \text{least}=|D-r|$$

- The image of the circle $S=0$ in a line is a circle of the same radius, centred at the image of the centre in that line.
- The greatest and least distance between two circles lying outside each other are $d+r_1+r_2$ and $d-r_1-r_2$.
- If one circle lies inside another, the greatest distance between them is $d+r_1+r_2$ and the least is $|r_1-r_2|-d$.

Read it along the line of centres. Every one of these extremes is attained on the straight line through the two centres, which is why the answers are always sums and differences.

30 QUICK FACTS WORTH MEMORISING (E·30)

i. S_1 is the power of the point and the square of the tangent length: two names for one number.
ii. The tangent at (x_1,y_1) is obtained from $S=0$ by the substitutions $x^2\to xx_1$, $y^2\to yy_1$, $2x\to x+x_1$, $2y\to y+y_1$.
iii. $T=0$ is the tangent, the chord of contact and the polar; which one it is depends only on where the point lies.
iv. $T=S_1$ is the chord bisected at the point, and $SS_1=T^2$ is the pair of tangents.
v. The equation of any circle has no xy term and equal coefficients of x^2 and y^2.
vi. Two circles are orthogonal when $2g_1g_2+2f_1f_2=c_1+c_2$; a circle cuts the axes orthogonally only in degenerate cases.
vii. A circle and a line meet in at most two points; a circle and a circle in at most two points.