[REVISION TOOL · JEE MAIN + ADVANCED · CLASS 11]

CONCEPT MAP

PERMUTATIONS & COMBINATIONS

5 TOPICS · 31 CORE IDEAS · ONE SHEET

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Ritesh Raj · M.Sc Mathematics, IIT Delhi
JEE Main · JEE Advanced · Olympiad Mathematics

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A COUNTING PRINCIPLES 01-06

01 THE TWO FUNDAMENTAL PRINCIPLES
A·01

MULTIPLICATION PRINCIPLE

If a task can be done in m ways and, in continuation, a second task can be done in n ways, then the two together can be done in $m \times n$ ways.

More generally, if a job has k parts and the job is finished only when every part is finished, the first part being possible in a_1 ways, the second in a_2 ways and so on, the whole job can be done in

$$a_1 \times a_2 \times a_3 \times \cdots \times a_k \text{ ways}$$

ADDITION PRINCIPLE

If one task can be done in m ways and another, mutually exclusive, task in n ways, then exactly one of the two can be done in $m+n$ ways.

AND means multiply, OR means add. Read the problem for the connecting word: stages that all happen multiply, alternatives that exclude one another add. Almost every counting error starts by getting this the wrong way round.

02 FACTORIAL NOTATION
A·02

For a positive integer n, the continued product of the first n natural numbers is called factorial n and written $n!$:

KEY

$$n! = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$$

i. $0! = 1$ and $1! = 1$.

ii. $n! = n\,(n-1)!$, which is how a factorial is unwound.

iii. $(2n)! = 2^n \cdot n! \cdot [1\cdot 3\cdot 5\cdots(2n-1)]$.

iv. $n!$ is defined only for non-negative integers; $(-3)!$ and $(1/2)!$ have no meaning here.

Why $0! = 1$. It is the empty product, and it is exactly the value that makes $n! = n(n-1)!$ and ${}^nC_0 = 1$ true at the boundary.

03 EXPONENT OF A PRIME IN $n!$
A·03

The highest power of a prime p that divides $n!$ is

KEY

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots$$

where [ ] is the greatest integer function. The sum stops as soon as $p^k > n$.

Trailing zeros. The number of zeros at the end of $n!$ is $E_5(n!)$, because factors of 5 are scarcer than factors of 2. So $100!$ ends in $20 + 4 = 24$ zeros.

04 PERMUTATIONS
A·04

A permutation is an arrangement of objects: first a selection, then an ordering. Order matters, so abc, acb, bac, bca, cab and cba are six different permutations of the same three letters.

KEY

$${}^nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)\cdots(n-r+1)$$

i. All n objects taken together: ${}^nP_n = n!$.

ii. ${}^nP_0 = 1$ and ${}^nP_1 = n$.

iii. ${}^nP_r = n \cdot {}^{n-1}P_{r-1}$.

05 COMBINATIONS
A·05

A combination is a selection. Here the order inside the group is irrelevant, so abc and cba are the same combination. The number of combinations of n different things taken r at a time is

KEY

$${}^nC_r = \frac{n!}{(n-r)!\,r!}$$

Every combination of r things can be ordered in $r!$ ways, which links the two counts:

$${}^nC_r = \frac{{}^nP_r}{r!} \implies {}^nP_r = {}^nC_r \cdot r!$$

Which one is it? If rearranging the chosen objects gives a genuinely different outcome, it is a permutation; if not, it is a combination. Committee members are a combination, a president and a secretary are a permutation.

06 KEY RESULTS ON nC{R}
A·06

i. ${}^nC_r = {}^nC_{n-r}$, for $0 \le r \le n$.

ii. ${}^nC_0 = {}^nC_n = 1$ and ${}^nC_1 = n$.

iii. If ${}^nC_x = {}^nC_y$, then either $x = y$ or $x + y = n$.

iv. ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$, for $1 \le r \le n$.

v. $r \cdot {}^nC_r = n \cdot {}^{n-1}C_{r-1}$.

vi. ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$.

vii. ${}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots = 2^{n-1}$.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

${}^{n-1}C_{r-1} + {}^{n-1}C_r = {}^nC_r$

Greatest value. ${}^nC_r$ is largest at $r = \frac{n}{2}$ when n is even, and at $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$, which are equal, when n is odd.

B ARRANGEMENTS 07-13

07 ARRANGEMENTS WITH REPETITION
B·07

If repetition is allowed, each of the r places may be filled by any of the n objects independently, so the number of arrangements of n different objects taken r at a time is

KEY

$$n^r$$

- Number of r-digit numbers from n given digits, repetition allowed: n^r.
- Number of ways of posting r letters into n letter-boxes: n^r.
- Number of ways r prizes can be given to n boys, a boy allowed any number: n^r.

Which is the exponent? The base is the number of choices available at each step and the exponent is the number of steps. Letters choose boxes, so it is $(\text{boxes})^{\text{letters}}$, not the other way round.

08 THINGS NOT ALL DIFFERENT
B·08

Suppose n objects are to be arranged, of which p are alike of one kind, q alike of a second kind and r alike of a third, with $p + q + r = n$. Then

KEY

$$\text{number of arrangements} = \frac{n!}{p!\,q!\,r!}$$

The same rule extends to any number of kinds; divide $n!$ by the factorial of the count of each repeated kind.

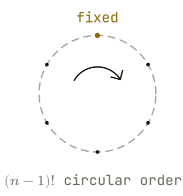
Why divide. Treating the alike objects as distinct would count every arrangement $p!\,q!\,r!$ times over, once for each internal shuffle that changes nothing.

09 CIRCULAR PERMUTATIONS
B·09

When objects are arranged round a circle only their relative positions matter, since the whole ring may be rotated. Fixing one object removes that freedom.

KEY

$$n \text{ different things round a circle} = (n-1)!$$



$(n-1)!$ circular orders

i. If clockwise and anticlockwise orders are not distinct, as with beads on a necklace or flowers in a garland, the count is $\frac{1}{2}(n-1)!$.

ii. n different things taken r at a time, clockwise and anticlockwise different: $\frac{{}^nP_r}{r}$.

iii. n different things taken r at a time, clockwise and anticlockwise the same: $\frac{{}^nP_r}{2r}$.

10 INCLUDED AND EXCLUDED OBJECTS
B·10

ALWAYS INCLUDED

The number of r-arrangements of n different things in which s specified things always appear is

KEY

$${}^{n-s}C_{r-s} \times r!$$

For a single specified thing this is the familiar $r \cdot {}^{n-1}P_{r-1}$.

NEVER INCLUDED

If one particular thing is never taken, only $n-1$ things are available:

$${}^{n-1}P_r$$

Two steps, always. Choose which objects go in, then arrange all r of them. Trying to do both at once is what makes these questions feel harder than they are.

11 TOGETHER AND NEVER TOGETHER
B·11

If m specified things must always come together, tie them into a single block. The block and the remaining $n-m$ things give $n-m+1$ objects to arrange, and the block can be shuffled internally:

KEY

$$m! \times (n-m+1)!$$

Subtracting from the unrestricted total gives the arrangements in which they never all come together:

$$n! - m!\,(n-m+1)!$$

TOGETHER, IN AN r-ARRANGEMENT

$$s! \times (r-s+1) \times {}^{n-s}P_{r-s}$$

Correction to the printed page. The source gives the never-together count as $n! - m!(n-m+1)$. The second term must be $m!\,(n-m+1)!$ — it is the together-count, and that carries a factorial.

12 NO TWO TOGETHER: THE GAP METHOD
B·12

To keep certain objects apart, arrange the others first and then drop the objects to be separated into the gaps between them.

↑ A ↑ B ↑ C ↑ D ↑

n items make $n+1$ gaps

If m objects of one kind are to be placed among n arranged objects so that no two of them are adjacent, choose m of the $n+1$ gaps:

KEY

$$n! \times {}^{n+1}P_m = n! \times {}^{n+1}C_m \times m!$$

The everyday case. Seating n boys and m girls in a row with no two girls together: $n!\,{}^{n+1}P_m$, which needs $m \le n+1$ or the arrangement is impossible.

13 RANK OF A WORD IN A DICTIONARY
B·13

To find the position of a word when all arrangements of its letters are listed in alphabetical order:

i. Write the letters in alphabetical order.

ii. Count the words starting with each letter that comes before the first letter of the given word; each contributes the number of arrangements of the remaining letters.

iii. Fix the first letter and repeat the count on the second letter, then the third, and so on.

iv. Add all those counts and add 1 for the word itself.

Repeated letters. When letters repeat, every block count uses $\frac{(\text{letters left})!}{\prod(\text{repeats})!}$ rather than a plain factorial. Forgetting this is the usual slip.

C SELECTIONS 14-20

14 RESTRICTED SELECTIONS
C·14

For combinations of n different things taken r at a time:

KEY

$$p \text{ particular things always included} = {}^{n-p}C_{r-p}$$

i. p particular things always excluded: ${}^{n-p}C_r$.

ii. p particular things never all together in the selection: ${}^nC_r - {}^{n-p}C_{r-p}$.

iii. No two of the chosen r consecutive, chosen from n in a row: ${}^{n-r+1}C_r$.

Include, then choose the rest. Fixing the p compulsory objects leaves $n-p$ objects from which $r-p$ more must come; the arithmetic collapses to a single symbol.

15 ALL POSSIBLE SELECTIONS: DISTINCT OBJECTS
C·15

When each of n different objects may independently be taken or left, the selections of zero or more objects number

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$$

Discarding the empty selection leaves the selections of at least one object:

$${}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n - 1$$

16 ALL POSSIBLE SELECTIONS: IDENTICAL OBJECTS
C·16

i. The number of ways of selecting r objects out of n identical objects is 1: only how many, never which.

ii. The total number of selections of zero or more objects from n identical objects is $n+1$.

iii. From a_1 alike of one kind, a_2 alike of a second kind and so on to a_n alike of the nth kind, the number of selections of at least one object is

KEY

$$(a_1+1)(a_2+1)\cdots(a_n+1) - 1$$

Where the -1 comes from. Each kind offers $a_i + 1$ choices, from taking none up to taking all a_i; the product counts every case including the one where nothing at all is taken, which is then removed.

17 IDENTICAL AND DISTINCT TOGETHER
C·17

Take n objects of which p are identical and the other $n-p$ are all different. The number of selections of r objects is

KEY

$${}^{n-p}C_r + {}^{n-p}C_{r-1} + \cdots + {}^{n-p}C_0, \quad \text{if } r \le p$$
$${}^{n-p}C_r + {}^{n-p}C_{r-1} + \cdots + {}^{n-p}C_{r-p}, \quad \text{if } r \ge p$$

and, for $a_1 + a_2 + \cdots + a_n + k$ objects in which a_i are alike of the ith kind and k are all distinct, the number of selections of at least one object is

$$(a_1+1)(a_2+1)\cdots(a_n+1)\cdot 2^k - 1$$

Reading the sum. Each term fixes how many of the identical objects are taken; the rest come from the distinct pile. The two cases differ only in where the sum has to stop.

18 THE COEFFICIENT METHOD
C·18

Selecting r things from n things of which p are alike of one kind, q alike of another, s alike of a third and so on is the coefficient of x^r in

KEY

$$(1 + x + x^2 + \cdots + x^p)$$
$$\times (1 + x + x^2 + \cdots + x^q)$$
$$\times (1 + x + x^2 + \cdots + x^s)\cdots$$

If every kind must be taken at least once, drop the constant term from each bracket; the answer is the coefficient of x^{r-n} in

$$(1 + x + \cdots + x^{p-1})$$
$$\times (1 + x + \cdots + x^{q-1})$$
$$\times (1 + x + \cdots + x^{s-1})\cdots$$

One bracket per kind. The exponent records how many of that kind were taken and the highest power records how many are available. Multiplying the brackets performs the whole count in one step.

19 SELECTIONS WITH REPETITION
C·19

If there are n kinds of object with an unlimited supply of each, the number of ways of selecting r objects, order not counting, is

KEY

$${}^{n+r-1}C_r = {}^{n+r-1}C_{n-1}$$

Do not confuse the two. With repetition and order counting the answer is n^r; with repetition and order not counting it is ${}^{n+r-1}C_r$. The second is the same count as distributing r identical objects among n kinds.

20 DIVISION INTO GROUPS
C·20

To divide $m+n+p$ different objects into three groups holding m, n and p objects, with m, n, p all different:

KEY

$$\frac{(m+n+p)!}{m!\,n!\,p!} \text{ if the order of the groups is not counted}$$

and $\frac{(m+n+p)! \times 3!}{m!\,n!\,p!}$ if it is. When the three groups are the same size, $m = n = p$, an extra $3!$ must be removed:

$$\frac{(3m)!}{(m!)^3\,3!} \text{ if unordered,} \qquad \frac{(3m)!}{(m!)^3} \text{ if ordered}$$

Equal sizes are the trap. When the groups have the same size they cannot be told apart, so every division has been counted $3!$ times. Unequal sizes label the groups by themselves and need no correction.

D DISTRIBUTIONS & EQUATIONS 21-25

21 DISTRIBUTING IDENTICAL OBJECTS
D·21

The number of ways of dividing n identical objects among r distinct groups is

KEY

$${}^{n+r-1}C_{r-1} \text{ if empty groups are allowed}$$

⋆ ⋆ ⋆ | ⋆ ⋆ | ⋆ ⋆ ⋆

$3+2+3=8$ into 3 boxes

and, if no group may be left empty,

$${}^{n-1}C_{r-1}$$

Bars between stars. Lay the n objects in a row and insert $r-1$ dividers. Allowing empty groups lets dividers share a gap, which is why the count changes.

22 INTEGRAL SOLUTIONS OF AN EQUATION
D·22

The number of solutions of

KEY

$$x_1 + x_2 + \cdots + x_r = n$$

i. in non-negative integers, $x_i \ge 0$: ${}^{n+r-1}C_{r-1}$.

ii. in positive integers, $x_i \ge 1$: ${}^{n-1}C_{r-1}$.

iii. with lower bounds $x_i \ge k_i$: substitute $y_i = x_i - k_i$ and use case 1 with n replaced by $n - \sum k_i$.

iv. with upper bounds as well: read off the coefficient of x^n in the product of the matching polynomials.

Same problem, three costumes. Distributing identical objects, selecting with repetition and solving this equation are one count wearing different clothes. Recognising that turns three techniques into one.

23 DISTRIBUTING DISTINCT OBJECTS
D·23

For m distinct objects placed into n distinct boxes, each object choosing a box independently, the count is n^m. This is also the number of functions from a set of m elements to a set of n elements.

KEY

$$\text{total} = n^m, \qquad \text{one-one} = {}^nP_m \ (m \le n)$$

If no box may be left empty, that is if the function is onto,

$$\sum_{k=0}^{n} (-1)^k\, {}^nC_k\, (n-k)^m$$

and when $m = n$ the onto functions are exactly the bijections, $n!$ of them.

24 NUMBER OF DIVISORS
D·24

Write $N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ with $p_1, \ldots, p_k$ distinct primes.

KEY

$$\text{number of divisors} = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_k+1)$$

i. Excluding 1 and N itself, subtract 2 from that product.

ii. Sum of all the divisors $= (p_1^0 + p_1^1 + \cdots + p_1^{\alpha_1})\cdots(p_k^0 + p_k^1 + \cdots + p_k^{\alpha_k})$.

iii. N can be written as a product of two factors in $\frac{1}{2}\prod(\alpha_i + 1)$ ways when N is not a perfect square, and $\frac{1}{2}[\prod(\alpha_i+1) + 1]$ ways when it is.

iv. A composite N splits into two coprime factors in 2^{k-1} ways, k being the number of distinct primes in N.

Why the product works. A divisor is built by choosing an exponent for each prime, anything from 0 to α_i; that is $\alpha_i + 1$ independent choices, so the multiplication principle finishes it.

25 INCLUSION AND EXCLUSION
D·25

To count the objects with at least one of several properties, add the single counts, subtract the pairs, add back the triples, and so on:

KEY

$$|A \cup B \cup C| = |A| + |B| + |C|$$
$$- |A \cap B| - |B \cap C| - |C \cap A|$$
$$+ |A \cap B \cap C|$$

The number with none of the properties is the total minus this.

Where it earns its keep. "At least one" conditions, counting integers divisible by none of a set of numbers, and onto functions are all this principle in different clothing.

E STANDARD PROBLEMS 26-31

26 POINTS, LINES AND TRIANGLES
E·26

Let n distinct points lie in a plane with no three collinear.

KEY

$$\text{line segments} = {}^nC_2, \qquad \text{triangles} = {}^nC_3$$

If m of those points are collinear, with $m \ge 3$, the collinear set contributes only one line and no triangles:

$$\text{lines} = {}^nC_2 - {}^mC_2 + 1, \qquad \text{triangles} = {}^nC_3 - {}^mC_3$$

POINTS ON A CIRCLE

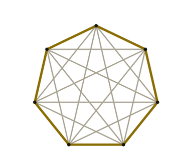
For n points on the circumference of a circle: straight lines ${}^nC_2$, triangles ${}^nC_3$, quadrilaterals ${}^nC_4$ and k-sided polygons ${}^nC_k$.

27 POLYGONS AND DIAGONALS
E·27

A convex polygon of n sides has ${}^nC_2$ lines through its vertices, of which n are sides, so

KEY

$$\text{diagonals} = {}^nC_2 - n = \frac{n(n-3)}{2}$$



$\frac{n(n-3)}{2}$ diagonals

i. Triangles with all three vertices at the polygon's vertices: ${}^nC_3$.

ii. Of those, the ones with no side common with the polygon: $\frac{n(n-4)(n-5)}{6}$.

iii. With exactly one side common: $n(n-4)$; with exactly two sides common: n.

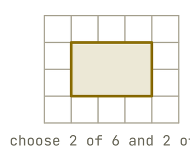
iv. Points where the diagonals cross inside, no three concurrent: ${}^nC_4$.

28 RECTANGLES AND SQUARES ON A GRID
E·28

A rectangle is fixed by choosing two of the horizontal lines and two of the vertical ones. On a grid of $m \times n$ unit squares there are $m+1$ and $n+1$ lines, so

KEY

$$\text{rectangles} = {}^{m+1}C_2 \times {}^{n+1}C_2$$



choose 2 of 6 and 2 of 5

ON AN $n \times n$ BOARD

$$\text{squares} = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

and the rectangles number $\left[\frac{n(n+1)}{2}\right]^2$, so the rectangles that are not squares are the difference of the two.

29 DERANGEMENTS
E·29

Any change in the given order of things is called a derangement. If n things are arranged in a row, the number of ways of rearranging them so that not one occupies its original place is

KEY

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

The first few values are $D_1 = 0$, $D_2 = 1$, $D_3 = 2$, $D_4 = 9$, $D_5 = 44$.

EXACTLY r DISPLACED

If exactly r objects are to leave their places and the other $n-r$ stay put, the total is

$${}^nC_{n-r} \cdot D_r = {}^nC_{n-r} \cdot r!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^r \frac{1}{r!}\right)$$

30 SUM OF THE NUMBERS FORMED
E·30

Take n distinct non-zero digits and form every n-digit number using each digit once. Each digit occupies each place $(n-1)!$ times, so

KEY

$$\text{sum} = (n-1)! \times (\text{sum of the digits}) \times \underbrace{111\ldots1}_{n \text{ ones}}$$

where the repunit equals $\frac{10^n - 1}{9}$.

If one digit is zero. Subtract the numbers that would begin with 0: the sum becomes $(n-1)!\,S\,R_n - (n-2)!\,S\,R_{n-1}$, with S the digit sum and R_k the repunit of k ones.

31 QUICK FACTS AND COMMON TRAPS
E·31

i. Arrange or select? Ask whether reordering the chosen objects changes the answer.

ii. AND multiplies, OR adds; "at least one" usually means total minus none.

iii. n^r counts repetition with order; ${}^{n+r-1}C_r$ counts repetition without order.

iv. Dividing into groups of equal size always needs a further division by the number of such groups factorial.

v. Round a table, fix one person first; on a necklace, halve the answer as well.

vi. When objects must be apart, arrange the others and use the gaps; when they must be together, tie them into a block.

vii. ${}^nC_r$ is an integer for every r, so a fractional answer means the count is wrong somewhere.

viii. Check small cases by hand. A formula that fails at $n = 2$ or $n = 3$ will fail everywhere.