[REVISION TOOL · JEE MAIN + ADVANCED · CLASS 11]

CONCEPT MAP

PARABOLA

5 TOPICS · 30 CORE IDEAS · ONE SHEET

SCAN TO PRACTICE

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REAL STUDY, GENUINE RESULTS.

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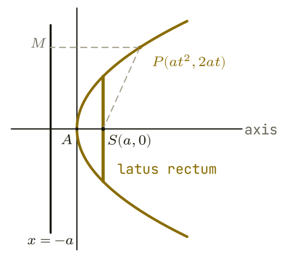
A THE CURVE & ITS FORMS 01–06

01 DEFINITION AND TERMINOLOGY
A·01

A parabola is the locus of a point moving in a plane so that its distance from a fixed point, the focus, equals its perpendicular distance from a fixed straight line, the directrix. Its eccentricity is therefore $e = 1$.

KEY

$$SP = PM$$



i. **Focal distance**: the distance of a point on the parabola from the focus.
ii. **Focal chord**: a chord that passes through the focus.
iii. **Double ordinate**: a chord perpendicular to the axis of symmetry.
iv. **Latus rectum**: the double ordinate through the focus, that is the focal chord perpendicular to the axis.
v. **Vertex**: the point where the conic meets its own axis.
vi. **Axis**: the line through the focus perpendicular to the directrix.

02 STANDARD FORM
A·02

Let S be the focus, QN the directrix and P any point of the curve. By the definition $PS = PN$, where PN is the perpendicular from P to QN, so $PS^2 = PN^2$:

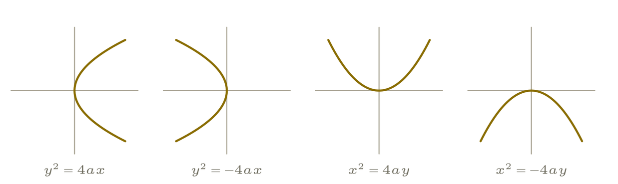
KEY

$$(x-a)^2 + y^2 = (a+x)^2 \implies y^2 = 4ax$$

The other standard forms are $y^2 = -4ax$, $x^2 = 4ay$ and $x^2 = -4ay$.

> **Reading the equation.** The squared variable names the axis: $y^2 = \dots$ opens along the X-axis, $x^2 = \dots$ along the Y-axis. The sign of the linear term says which way it opens.

03 THE FOUR STANDARD FORMS
A·03



AXIS ALONG X

	$y^2 = 4ax$	$y^2 = -4ax$
Vertex	$(0,0)$	$(0,0)$
Focus	$(a,0)$	$(-a,0)$
Directrix	$x=-a$	$x=a$
Axis	$y=0$	$y=0$
Latus rectum	$x=a$	$x=-a$
Parametric	$(at^2, 2at)$	$(-at^2, 2at)$

AXIS ALONG Y

	$x^2 = 4ay$	$x^2 = -4ay$
Vertex	$(0,0)$	$(0,0)$
Focus	$(0,a)$	$(0,-a)$
Directrix	$y=-a$	$y=a$
Axis	$x=0$	$x=0$
Latus rectum	$y=a$	$y=-a$
Parametric	$(2at, at^2)$	$(2at, -at^2)$

In every case the length of the latus rectum is $4a$ and the eccentricity is 1.

> **Correction to the printed page.** The source gives the parametric point of $x^2 = 4ay$ as $(2at^2, at^2)$. It must be $(2at, at^2)$, since then $x^2 = 4a^2t^2 = 4a \cdot at^2 = 4ay$.

04 PARAMETRIC EQUATIONS
A·04

The pair $x = at^2$, $y = 2at$ satisfies $y^2 = 4ax$ for every real t, so they are the parametric equations of the parabola, with t the parameter.

KEY

$$x = at^2, \qquad y = 2at, \qquad t \in R$$

The point $(at^2, 2at)$ lies on $y^2 = 4ax$ for all real t and is usually called simply the point t.

> **Why it is worth using.** One parameter replaces two coordinates tied by a constraint, so chord, tangent and normal conditions become polynomial relations in t_1 and t_2 that can be solved by inspection.

05 GENERAL PARABOLA
A·05

Shifting the vertex to (h,k) and keeping the axis parallel to the X-axis gives

KEY

$$(y-k)^2 = 4a(x-h)$$

and with the axis parallel to the Y-axis, $(x-h)^2 = 4a(y-k)$, which is the same as $y = Ax^2 + Bx + C$ with $A \neq 0$.

WHEN IS A CONIC A PARABOLA?

The general second-degree equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a parabola when it is non-degenerate and

$$h^2 = ab$$

> **Locating the vertex quickly.** For $y = Ax^2 + Bx + C$ the axis is $x = -\dfrac{B}{2A}$ and the latus rectum is $\dfrac{1}{|A|}$.

06 FOCAL DISTANCE AND FOCAL CHORD
A·06

For $y^2 = 4ax$ the focal distance of $P(x_1, y_1)$ is its distance from the directrix:

KEY

$$SP = x_1 + a = a(1+t^2)$$

LENGTH OF A FOCAL CHORD

If the focal chord makes an angle θ with the axis, its length is

$$4a \operatorname{cosec}^2\theta$$

so the shortest focal chord is the latus rectum, of length $4a$, at $\theta = 90°$.

> **The harmonic property.** If a focal chord is split by the focus into lengths ℓ_1 and ℓ_2, then $\dfrac{1}{\ell_1} + \dfrac{1}{\ell_2} = \dfrac{1}{a}$: the semi-latus rectum $2a$ is the harmonic mean of the two focal segments.

B CHORDS 07–12

07 EQUATION OF A CHORD
B·07

The chord joining (x_1, y_1) and (x_2, y_2) on $y^2 = 4ax$ is

$$y(y_1 + y_2) = 4ax + y_1y_2$$

and in parameters, joining the points t_1 and t_2,

KEY

$$y(t_1 + t_2) = 2(x + at_1t_2)$$

> **Slope of a chord.** From the parametric form the slope is $\dfrac{2}{t_1+t_2}$, which is why the sum $t_1 + t_2$ appears in every direction condition and the product t_1t_2 in every position condition.

08 LENGTH OF A CHORD
B·08

The chord cut on $y^2 = 4ax$ by the line $y = mx + c$ has length

KEY

$$\frac{4}{m^2}\sqrt{a(1+m^2)(a-mc)}$$

> **Read off the three cases.** The line cuts the parabola in two points, touches it, or misses it according as $a - mc$ is positive, zero or negative. The middle case is the tangency condition $c = a/m$.

09 FOCAL CHORD IN PARAMETERS
B·09

The chord joining the points t_1 and t_2 passes through the focus if and only if

KEY

$$t_1t_2 = -1$$

The length of that focal chord is

$$a(t_2 - t_1)^2 = a\left(t_1 + \frac{1}{t_1}\right)^2$$

> **Minimum length.** Since $\left|t_1 + \dfrac{1}{t_1}\right| \geq 2$, the focal chord is never shorter than $4a$, the latus rectum, with equality at $t_1 = \pm 1$.

10 CHORD WITH A GIVEN MID-POINT
B·10

The chord of $y^2 = 4ax$ whose middle point is $P(x_1, y_1)$ is $T = S_1$, that is

KEY

$$yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

where $T \equiv yy_1 - 2a(x+x_1)$ and $S_1 \equiv y_1^2 - 4ax_1$.

> **Its slope.** Rearranged, the chord has slope $\dfrac{2a}{y_1}$, so a chord bisected at a given point has a direction fixed by the ordinate of that point alone.

11 DIAMETER OF A PARABOLA
B·11

The locus of the mid-points of a system of parallel chords of slope m is called a diameter of the parabola. For $y^2 = 4ax$ it is the line

KEY

$$y = \frac{2a}{m}$$

> **Always parallel to the axis.** Every diameter of a parabola is a straight line parallel to its axis, and the tangent at the point where a diameter meets the curve is parallel to the chords it bisects.

12 POSITION OF A POINT
B·12

The point (x_1, y_1) lies outside, on, or inside the parabola $y^2 = 4ax$ according as

$$S_1 = y_1^2 - 4ax_1 \text{ is } > 0,\ = 0 \text{ or } < 0$$

NUMBER OF TANGENTS

i. From a point outside the parabola, two tangents can be drawn.
ii. From a point on the parabola, exactly one tangent can be drawn.
iii. From a point inside the parabola, no real tangent can be drawn.

C TANGENTS 13–19

13 CONDITION OF TANGENCY
C·13

The line $y = mx + c$ touches the parabola $y^2 = 4ax$ if and only if

KEY

$$c = \frac{a}{m}$$

and the point of contact is then

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$$

> **Correction to the printed page.** The source heads this section "condition of tendency"; the word is **tangency**.

14 EQUATION OF THE TANGENT
C·14

POINT FORM

At the point $P(x_1, y_1)$ of $y^2 = 4ax$ the tangent is $T = 0$:

$$yy_1 = 2a(x + x_1)$$

SLOPE FORM

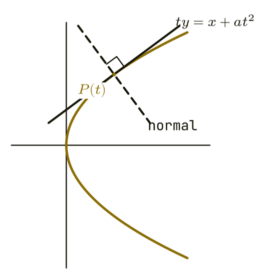
KEY

$$y = mx + \frac{a}{m}, \qquad \text{contact at } \left(\frac{a}{m^2}, \frac{2a}{m}\right)$$

PARAMETRIC FORM

At the point $(at^2, 2at)$ the tangent is

$$ty = x + at^2$$



15 POINT OF INTERSECTION OF TANGENTS
C·15

The tangents at the points $P(t_1)$ and $Q(t_2)$ meet at

KEY

$$T(at_1t_2,\ a(t_1 + t_2))$$

i. Its x-coordinate is the geometric mean of the x-coordinates of P and Q.
ii. Its y-coordinate is the arithmetic mean of the y-coordinates of P and Q.

> **Tangents at the ends of a focal chord.** There $t_1t_2 = -1$, so the meeting point is $(-a,\ a(t_1+t_2))$ — it lies on the directrix, and the two tangents are perpendicular.

16 PAIR OF TANGENTS
C·16

The pair of tangents drawn from an external point $P(x_1, y_1)$ to $y^2 = 4ax$ is

KEY

$$SS_1 = T^2$$

where

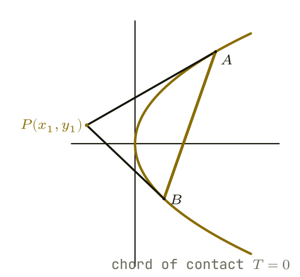
$$S = y^2 - 4ax$$
$$S_1 = y_1^2 - 4ax_1$$
$$T = yy_1 - 2a(x + x_1)$$

17 CHORD OF CONTACT
C·17

The chord joining the two points where the tangents from $P(x_1, y_1)$ touch $y^2 = 4ax$ is $T = 0$:

KEY

$$yy_1 = 2a(x + x_1)$$



> **Same symbol, three jobs.** $T = 0$ is the tangent when P is on the curve, the chord of contact when P is outside it, and the polar of P in every case.

18 DIRECTOR CIRCLE
C·18

The locus of the point of intersection of perpendicular tangents to $y^2 = 4ax$ is called the director circle of the parabola. Its equation is

KEY

$$x + a = 0$$

which is the directrix itself, so the director circle of a parabola degenerates into a straight line.

> **Why a line.** A circle of infinite radius is a line; as an ellipse flattens into a parabola its director circle grows without bound and in the limit becomes the directrix.

19 STANDARD TANGENT RESULTS
C·19

i. The slope of the tangent at the point $P(t)$ is $\dfrac{1}{t}$.
ii. The tangent at the vertex is the Y-axis, $x = 0$.
iii. The foot of the perpendicular from the focus to any tangent lies on the tangent at the vertex.
iv. The tangent at any point bisects the angle between the focal radius SP and the perpendicular from P to the directrix.
v. The subtangent at $P(x_1, y_1)$ is $2x_1$, twice the abscissa of the point.
vi. Tangents drawn at the extremities of the latus rectum intersect at right angles, on the directrix.

> **The circle on a focal radius.** The circle described on any focal radius as diameter touches the tangent at the vertex — a restatement of result 3.

D NORMALS 20–25

20 EQUATION OF THE NORMAL
D·20

POINT FORM

At (x_1, y_1) on $y^2 = 4ax$ the normal is

$$y - y_1 = \frac{-y_1}{2a}(x - x_1)$$

PARAMETRIC FORM

At the point $(at^2, 2at)$,

KEY

$$y + tx = 2at + at^3$$

SLOPE FORM

$$y = mx - 2am - am^3, \qquad \text{foot at } (am^2, -2am)$$

21 CONDITION FOR A NORMAL
D·21

The line $y = mx + c$ is a normal to the parabola $y^2 = 4ax$ if and only if

KEY

$$c = -2am - am^3$$

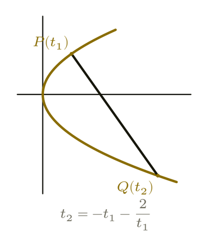
> **Compare the two conditions.** Tangency needs $c = \dfrac{a}{m}$ and normality needs $c = -2am - am^3$. Reading a problem for which of the two is being imposed is half the work.

22 NORMAL CHORD
D·22

If the normal at $P(t_1)$ meets the parabola again at $Q(t_2)$, the two parameters are connected by

KEY

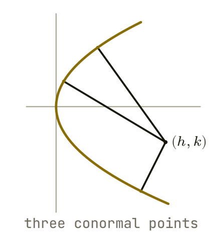
$$t_2 = -t_1 - \frac{2}{t_1}$$



> **Shortest normal chord.** Since $|t_2| = \left|t_1 + \dfrac{2}{t_1}\right| \geq 2\sqrt{2}$, the normal chord is shortest when $t_1 = \pm\sqrt{2}$, and its least length is $6\sqrt{3}\,a$.

23 CONORMAL POINTS
D·23

The points of a parabola at which the normals drawn are concurrent, that is they all pass through one point, are called conormal points.



Substituting (h,k) into the slope form gives the cubic in m

$$am^3 + (2a - h)m + k = 0$$

whose three roots are the slopes of the three normals from (h,k).

> **How many are real.** If the three normals drawn to $y^2 = 4ax$ from a point (h,k) are all real, then $h > 2a$.

24 KEY RESULTS ON CONORMAL POINTS
D·24

i. The sum of the slopes of the normals at the conormal points is zero: $m_1 + m_2 + m_3 = 0$.
ii. The sum of the ordinates of the conormal points is zero, since it equals $-2a(m_1 + m_2 + m_3) = 0$.
iii. The centroid of the triangle formed by the conormal points lies on the axis of the parabola.

The vertices of that triangle are $(am_1^2, -2am_1)$, $(am_2^2, -2am_2)$ and $(am_3^2, -2am_3)$; because the ordinates sum to zero the centroid falls on the X-axis.

KEY

$$\text{centroid} = \left(\frac{a(m_1^2 + m_2^2 + m_3^2)}{3},\ 0\right)$$

25 STANDARD NORMAL RESULTS
D·25

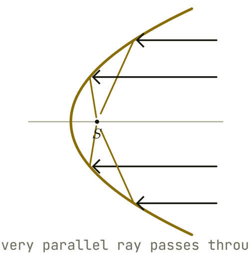
i. The slope of the normal at the point $P(t)$ is $-t$.
ii. The normal at any point passes through the focus only when that point is the vertex.
iii. The subnormal is constant and equal to $2a$, the semi-latus rectum, at every point of the parabola.
iv. If the normal at $P(t_1)$ subtends a right angle at the vertex, then $t_1^2 = 2$.
v. The normal chord at a point whose ordinate equals its abscissa subtends a right angle at the focus.

> **The constant subnormal.** No other conic has this property; it is often the fastest way to recognise a parabola from a differential-equation problem.

E STANDARD RESULTS 26–30

26 REFLECTION PROPERTY
E·26

A ray travelling parallel to the axis of a parabola is reflected at the curve straight through the focus, and conversely every ray from the focus leaves parallel to the axis.



The proof is the tangent result of section 19: the tangent at P bisects the angle between SP and the line through P parallel to the axis, so the angles of incidence and reflection are equal.

> **Why dishes are parabolic.** Satellite dishes, headlamp mirrors and solar concentrators are all surfaces of revolution of a parabola, for exactly this reason.

27 CIRCLES AND THE PARABOLA
E·27

i. The circle described on any focal chord as diameter touches the directrix.
ii. The circle described on any focal radius as diameter touches the tangent at the vertex.
iii. The circle through the focus and the two points where a normal chord meets the curve has its centre on the axis.
iv. The circle through three points of the parabola with parameters t_1, t_2, t_3 meets it again at t_4 with $t_1 + t_2 + t_3 + t_4 = 0$.

> **Reading result 4.** A circle and a parabola meet in at most four points, and their parameters always sum to zero — which turns many circle-parabola questions into one linear equation.

28 RESULTS IN THE PARAMETER T
E·28

i. The slope of the tangent at $P(t)$ is $\dfrac{1}{t}$; the slope of the normal is $-t$.
ii. If $P(t_1)$ and $Q(t_2)$ are the ends of a focal chord, then $t_1t_2 = -1$.
iii. If the normal at $P(t_1)$ meets the curve again at $Q(t_2)$, then $t_2 = -t_1 - \dfrac{2}{t_1}$.
iv. The tangents at $P(t_1)$ and $Q(t_2)$ meet at $(at_1t_2,\ a(t_1 + t_2))$.
v. Tangents drawn at the extremities of the latus rectum intersect at right angles.
vi. If $P(t_1)$ and $Q(t_2)$ are two points of the parabola and PQ subtends a right angle at the vertex, then $t_1t_2 = -4$.

29 STANDARD LOCI
E·29

i. The foot of the perpendicular from the focus to a variable tangent traces the tangent at the vertex, $x = 0$.
ii. The point of intersection of perpendicular tangents traces the directrix, $x + a = 0$.
iii. The mid-points of a family of parallel chords of slope m trace the line $y = \dfrac{2a}{m}$.

and the mid-points of all focal chords of $y^2 = 4ax$ trace the parabola

KEY

$$y^2 = 2a(x - a)$$

> **A parabola inside a parabola.** That locus is itself a parabola of the same latus rectum halved, with vertex at the focus of the original.

30 QUICK FACTS AND COMMON TRAPS
E·30

i. Eccentricity is always 1; a parabola has one focus, one directrix and no centre.
ii. $S_1 > 0$ means outside, $S_1 < 0$ means inside — the opposite sign convention to what many students expect.
iii. $T = 0$ is the tangent, the chord of contact and the polar; $T = S_1$ is the chord bisected at the point; $SS_1 = T^2$ is the pair of tangents.
iv. The tangency condition $c = \dfrac{a}{m}$ and the normality condition $c = -2am - am^3$ are easy to interchange by accident.
v. The subnormal is constant at $2a$, but the subtangent $2x_1$ is not.
vi. Three normals can be drawn from a point, at most; from a point with $h \leq 2a$ only one of them is real.
vii. Always check whether the given parabola is in standard position; if not, shift the origin to the vertex before quoting any formula.