[REVISION TOOL · JEE MAIN + ADVANCED · CLASS 11]

CONCEPT MAP

ELLIPSE

5 TOPICS · 31 CORE IDEAS · ONE SHEET

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REAL STUDY, GENUINE RESULTS.

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A THE CURVE & ITS ELEMENTS 01-07

01 DEFINITION AND STANDARD EQUATION

A·01

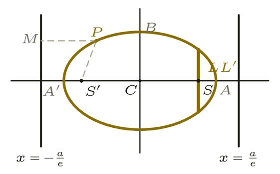
An ellipse is the locus of a point moving in a plane so that the ratio of its distance from a fixed point, the focus, to its distance from a fixed line, the directrix, is a constant less than one. That constant is the eccentricity e.

KEY

$$\frac{SP}{PM} = e\ (e<1), \qquad \text{so} \qquad SP = e\,PM$$

Taking the centre at the origin and the major axis along X, this reduces to the standard equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



For $a>b$ the foci are $S(ae,0)$ and $S'(-ae,0)$, the directrices are $x=\pm\frac{a}{e}$, and

$$b^2=a^2(1-e^2)$$

02 USEFUL TERMS

A·02

i. **Foci**: the two fixed points S and S', at $(\pm ae, 0)$.

ii. **Directrices**: the lines $x=\frac{a}{e}$ and $x=-\frac{a}{e}$; the distance between them is $\frac{2a}{e}$.

iii. **Axes**: AA' is the major axis of length $2a$ and BB' the minor axis of length $2b$.

iv. **Centre**: the point C where the axes cross. Every chord through C is bisected at C.

v. **Vertices**: the ends $A(a,0)$ and $A'(-a,0)$ of the major axis.

vi. **Focal chord**: a chord through a focus.

vii. **Double ordinate**: a chord perpendicular to the major axis; half of it is the ordinate.

03 LATUS RECTUM

A·03

The latus rectum is the focal chord perpendicular to the major axis. There are two of them, at $x=\pm ae$, and each has length

$$LL' = \frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}}$$

Equivalently,

$$LL' = 2a(1-e^2) = 2e\left(\frac{a}{e}-ae\right)$$

which is twice the eccentricity times the distance from a focus to the corresponding directrix. The ends of the latus rectum are $\left(\pm ae,\ \pm\frac{b^2}{a}\right)$.

04 FOCAL DISTANCES

A·04

For a point $P(x,y)$ on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $a>b$, the two focal distances are

KEY

$$SP = a-ex, \qquad S'P = a+ex$$

so their sum is constant:

$$SP+S'P = 2a = \text{length of the major axis}$$

The string construction. This constant-sum property is the second definition of an ellipse, and it is what lets you draw one with a loop of string round two pins.

05 THE TWO ORIENTATIONS

A·05

	$a>b$	$a<b$
Centre	$(0,0)$	$(0,0)$
Vertices	$(\pm a,0)$	$(0,\pm b)$
Foci	$(\pm ae,0)$	$(0,\pm be)$
Major axis	$2a,\ y=0$	$2b,\ x=0$
Minor axis	$2b,\ x=0$	$2a,\ y=0$
Directrices	$x=\pm\frac{a}{e}$	$y=\pm\frac{b}{e}$
Eccentricity	$\sqrt{1-\frac{b^2}{a^2}}$	$\sqrt{1-\frac{a^2}{b^2}}$
Latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Focal distances	$a\pm ex$	$b\pm ey$

Correction to the printed page. The source lists the vertices for $a<b$ as $(0,-b)$ and $(0,-b)$. They are $(0,b)$ and $(0,-b)$.

06 AUXILIARY CIRCLE AND ECCENTRIC ANGLE

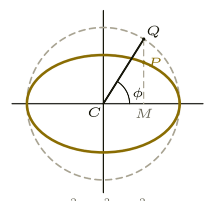
A·06

The circle described on the major axis as diameter is the auxiliary circle of the ellipse:

KEY

$$x^2+y^2=a^2$$

Let P be a point of the ellipse. Draw PM perpendicular to the major axis and produce it to meet the auxiliary circle at Q. The angle $\angle XCQ=\phi$ is called the eccentric angle of P.



$x^2+y^2=a^2$

The stretch. Q is $(a\cos\phi,\ a\sin\phi)$ and P is $(a\cos\phi,\ b\sin\phi)$, so the ellipse is the auxiliary circle with every ordinate shrunk in the ratio $b:a$.

07 PARAMETRIC EQUATIONS

A·07

Taken together, the equations

KEY

$$x=a\cos\phi, \qquad y=b\sin\phi$$

are the parametric equations of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, with the eccentric angle ϕ as parameter. The point is written simply as ϕ, or as $P(\phi)$.

CENTRE ELSEWHERE

If the centre is at (x_1,y_1) and the axes stay parallel to the coordinate axes,

$$\frac{(x-x_1)^2}{a^2}+\frac{(y-y_1)^2}{b^2}=1$$

Is a conic an ellipse? $ax^2+2hxy+by^2+2gx+2fy+c=0$ is an ellipse when it is non-degenerate and $h^2<ab$; it is a circle in the special case $a=b,\ h=0$.

B POINT, LINE & CHORD 08-13

08 POSITION OF A POINT

B·08

Write S_1 for the value of $S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$ at the point $P(x_1,y_1)$.

KEY

$$S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$$

i. $S_1>0$: P lies outside the ellipse.

ii. $S_1=0$: P lies on the ellipse.

iii. $S_1<0$: P lies inside the ellipse.

How many tangents. Two tangents can be drawn from P, and they are real and distinct, coincident, or imaginary according as P lies outside, on, or inside the ellipse.

09 A LINE AND AN ELLIPSE

B·09

Substituting $y=mx+c$ into the ellipse gives a quadratic in x whose discriminant decides everything. The line meets the ellipse in two points, touches it, or misses it according as

KEY

$$c^2<a^2m^2+b^2, \quad c^2=a^2m^2+b^2, \quad c^2>a^2m^2+b^2$$

CONDITION OF TANGENCY

$$c^2=a^2m^2+b^2$$

One condition, two tangents. Squaring gives $c=\pm\sqrt{a^2m^2+b^2}$, which is why every direction has exactly two tangents, one on each side of the centre.

10 EQUATION OF A CHORD

B·10

If $P(a\cos\theta,\ b\sin\theta)$ and $Q(a\cos\phi,\ b\sin\phi)$ are two points of the ellipse, the chord joining them is

KEY

$$\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta-\phi}{2}\right)$$

Letting $\phi\to\theta$. The right-hand side becomes 1 and the chord becomes the tangent at θ. Every tangent formula on this sheet is that limit.

11 CHORD WITH A GIVEN MID-POINT

B·11

The chord of the ellipse whose middle point is (x_1,y_1) is $T=S_1$, that is

KEY

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

Its slope. Rearranged, the chord has slope $-\frac{b^2x_1}{a^2y_1}$, and the line joining the centre to the mid-point has slope $\frac{y_1}{x_1}$; their product is $-\frac{b^2}{a^2}$, which is the conjugate-diameter condition of section 13.

12 FOCAL CHORD IN ECCENTRIC ANGLES

B·12

If α and β are the eccentric angles of the ends of a focal chord of the ellipse, then

KEY

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{e-1}{e+1} \quad \text{or} \quad \frac{e+1}{e-1}$$

according as the chord passes through the focus $(ae,0)$ or $(-ae,0)$.

Semi-latus rectum. If a focal chord is split by the focus into ℓ_1 and ℓ_2, then $\frac{1}{\ell_1}+\frac{1}{\ell_2}=\frac{2a}{b^2}$: the semi-latus rectum is the harmonic mean of the two focal segments.

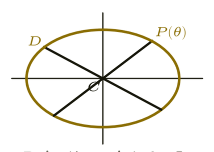
13 DIAMETER AND CONJUGATE DIAMETERS

B·13

The locus of the mid-points of a system of parallel chords of slope m is a diameter of the ellipse:

KEY

$$y=-\frac{b^2}{a^2m}x$$



D is the point $\theta+\frac{\pi}{2}$

Two diameters are conjugate when each bisects the chords parallel to the other. Their slopes satisfy

$$m_1m_2=-\frac{b^2}{a^2}$$

In eccentric angles. If CP and CD are conjugate semi-diameters, the eccentric angles of P and D differ by $\frac{\pi}{2}$: if P is θ, then D is $\theta+\frac{\pi}{2}$.

C TANGENTS 14-20

14 EQUATION OF THE TANGENT

C·14

POINT FORM

The tangent at (x_1,y_1) is $T=0$:

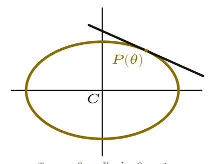
$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

PARAMETRIC FORM

At the point $(a\cos\theta,\ b\sin\theta)$,

KEY

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$



$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$

SLOPE FORM

$$y=mx\pm\sqrt{a^2m^2+b^2}$$

touching at the points

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}},\ \mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$$

15 POINT OF INTERSECTION OF TANGENTS

C·15

The tangents at the points θ and ϕ of the ellipse meet at

KEY

$$\left(a\,\frac{\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}},\ b\,\frac{\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}\right)$$

A check worth doing. Put $\phi=\theta$ and the point collapses to $(a\cos\theta,\ b\sin\theta)$, the point of contact itself, as it must.

16 PAIR OF TANGENTS

C·16

The combined equation of the two tangents drawn from $P(x_1,y_1)$ to the ellipse is

KEY

$$SS_1=T^2$$

written out,

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$$

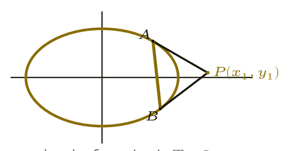
17 CHORD OF CONTACT

C·17

The chord joining the two points where the tangents from (x_1,y_1) touch the ellipse is $T=0$:

KEY

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$



chord of contact $T=0$

Same symbol, three jobs. $T=0$ is the tangent when P lies on the ellipse, the chord of contact when P lies outside it, and the polar of P in every case.

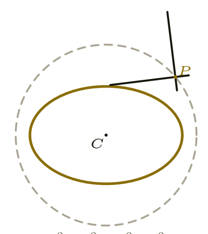
18 DIRECTOR CIRCLE

C·18

The locus of the point of intersection of perpendicular tangents to an ellipse is called its director circle. Its equation is

KEY

$$x^2+y^2=a^2+b^2$$



$x^2+y^2=a^2+b^2$

It is a circle concentric with the ellipse, of radius $\sqrt{a^2+b^2}$.

Where it comes from. Impose $c^2=a^2m^2+b^2$ on both tangents through (h,k) and set the product of the two slopes equal to -1; the condition collapses to $h^2+k^2=a^2+b^2$.

19 POLE AND POLAR

C·19

For any point $P(x_1,y_1)$, inside or outside, the line $T=0$ is called the polar of P with respect to the ellipse, and P is the pole of that line.

KEY

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

i. The pole of $lx+my+n=0$ is $\left(-\frac{a^2l}{n},\ -\frac{b^2m}{n}\right)$.

ii. If the polar of P passes through Q, then the polar of Q passes through P; such points are conjugate.

iii. The polar of a focus is the corresponding directrix.

20 STANDARD TANGENT RESULTS

C·20

i. The product of the lengths of the perpendiculars from the two foci to any tangent is b^2.

ii. The feet of those perpendiculars lie on the auxiliary circle $x^2+y^2=a^2$.

iii. If the tangent at P meets a directrix at F, then PF subtends a right angle at the corresponding focus, that is $\angle PSF=\frac{\pi}{2}$.

iv. The tangent at P bisects the external angle between the focal radii SP and $S'P$.

v. The portion of a tangent between the point of contact and the directrix subtends a right angle at the focus.

Correction to the printed page. The source spells it "auxilliary"; the word is **auxiliary**.

D NORMALS 21-26

21 EQUATION OF THE NORMAL

D·21

POINT FORM

The normal at (x_1,y_1) to the ellipse is

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$$

PARAMETRIC FORM

At the point $(a\cos\theta,\ b\sin\theta)$,

KEY

$$ax\sec\theta-by\,\mathrm{cosec}\,\theta=a^2-b^2$$

SLOPE FORM

$$y=mx\mp\frac{m(a^2-b^2)}{\sqrt{a^2+b^2m^2}}$$

at the points $\left(\mp\frac{a^2}{\sqrt{a^2+b^2m^2}},\ \mp\frac{b^2m}{\sqrt{a^2+b^2m^2}}\right)$.

22 CONDITION FOR A NORMAL

D·22

The line $y=mx+c$ is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if and only if

KEY

$$c^2=\frac{m^2(a^2-b^2)^2}{a^2+b^2m^2}$$

Do not mix them up. Tangency asks for $c^2=a^2m^2+b^2$; normality asks for the expression above. Deciding which the question imposes is most of the work.

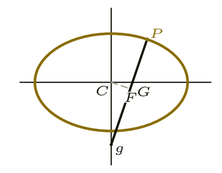
23 THE NORMAL AND THE AXES

D·23

Let the normal at P meet the major axis at G and the minor axis at g, and let CF be the perpendicular from the centre onto that normal. Then

KEY

$$PF\cdot PG=b^2, \qquad PF\cdot Pg=a^2$$



i. $PG\cdot Pg=SP\cdot S'P$.

ii. $CG\cdot CT=CS^2$, where T is the point at which the tangent at P meets the major axis.

iii. The normal at P bisects the internal angle between the focal radii SP and $S'P$.

24 CONORMAL POINTS

D·24

From a point on a given ellipse exactly one normal can be drawn. From a point not on the ellipse, infinitely many lines pass through it and cut the curve, but at most four of them are normals at the points where they cut. Those points of the ellipse are called conormal points.

Four, not three. The parabola has three conormal points and the ellipse four, because the normal condition is a cubic there and a quartic here.

25 ECCENTRIC ANGLES OF CONORMAL POINTS

D·25

i. The sum of the eccentric angles of the conormal points is an odd multiple of π.

ii. If $\theta_1,\theta_2,\theta_3,\theta_4$ are the eccentric angles of four points whose normals are concurrent, then $\Sigma\cos(\theta_1+\theta_2)=0$ and $\Sigma\sin(\theta_1+\theta_2)=0$.

If the normals at the four points $P(x_1,y_1)$, $Q(x_2,y_2)$, $R(x_3,y_3)$ and $S(x_4,y_4)$ of the ellipse are concurrent, then

KEY

$$(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4$$

26 CONCURRENCY OF THREE NORMALS

D·26

If the normals at $P(x_1,y_1)$, $Q(x_2,y_2)$ and $R(x_3,y_3)$ on the ellipse are concurrent, then

$$\begin{vmatrix} x_1 & y_1 & x_1y_1 \\ x_2 & y_2 & x_2y_2 \\ x_3 & y_3 & x_3y_3 \end{vmatrix}=0$$

and, if the same three points are given by their eccentric angles $P(\alpha)$, $Q(\beta)$ and $R(\gamma)$, then

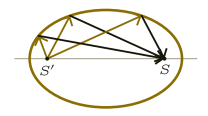
$$\begin{vmatrix} \sec\alpha & \mathrm{cosec}\,\alpha & 1 \\ \sec\beta & \mathrm{cosec}\,\beta & 1 \\ \sec\gamma & \mathrm{cosec}\,\gamma & 1 \end{vmatrix}=0$$

E STANDARD RESULTS 27-31

27 REFLECTION PROPERTY

E·27

A ray of light issuing from one focus of an ellipse is reflected at the curve straight through the other focus. This follows from the tangent result of section 20: the normal at P bisects the angle SPS', so the angles of incidence and reflection are equal.



every ray from S' reaches S

Whispering galleries. An elliptical room reflects every sound wave from one focus to the other, which is why a whisper at one focus is heard clearly at the other and nowhere in between.

28 CONJUGATE DIAMETER RESULTS

E·28

Let CP and CD be a pair of conjugate semi-diameters of the ellipse, with P at the eccentric angle θ and D at $\theta+\frac{\pi}{2}$.

i. $CP^2+CD^2=a^2+b^2$: the sum of the squares of conjugate semi-diameters is constant.

ii. The tangent at P is parallel to CD, and the tangent at D is parallel to CP.

iii. The parallelogram formed by the tangents at the ends of a pair of conjugate diameters has constant area $4ab$.

iv. The equi-conjugate diameters, for which $CP=CD$, are $y=\pm\frac{b}{a}x$, each of length $\sqrt{2(a^2+b^2)}$.

v. If CP and CD are conjugate, then $SP\cdot S'P=CD^2$.

29 STANDARD LOCI AND AREA

E·29

i. The feet of the perpendiculars from the foci to a variable tangent trace the auxiliary circle $x^2+y^2=a^2$.

ii. The point of intersection of perpendicular tangents traces the director circle $x^2+y^2=a^2+b^2$.

iii. The mid-points of a family of parallel chords of slope m trace the diameter $y=-\frac{b^2}{a^2m}x$.

The area enclosed by the ellipse is

KEY

$$\pi ab$$

Why πab. Stretching the auxiliary circle by the factor b/a in the y-direction turns area πa^2 into $\pi a^2\cdot\frac{b}{a}=\pi ab$.

30 SOME IMPORTANT RESULTS

E·30

The product of the lengths of the perpendicular segments from the foci to any tangent is b^2, and the feet of those perpendiculars lie on the auxiliary circle.

KEY

$$SY\cdot S'Y'=b^2$$

i. If the tangent at P meets the directrix at F, then $\angle PSF=\frac{\pi}{2}$ at the corresponding focus.

ii. The circle on any focal chord as diameter cannot cut the directrix.

iii. The locus of the mid-points of focal chords is a similar, smaller ellipse.

iv. The tangent and the normal at any point bisect the external and internal angles between the focal radii.

31 QUICK FACTS AND COMMON TRAPS

E·31

i. Always check whether $a>b$ or $a<b$ first; the foci, directrices and latus rectum all swap axes.

ii. $b^2=a^2(1-e^2)$ holds only when $a>b$; otherwise it is $a^2=b^2(1-e^2)$.

iii. $S_1>0$ means outside and $S_1<0$ means inside — the same convention as the circle and the parabola.

iv. $T=0$ is the tangent, the chord of contact and the polar; $T=S_1$ is the chord bisected at the point; $SS_1=T^2$ is the pair of tangents.

v. Tangency needs $c^2=a^2m^2+b^2$; normality needs $c^2=\frac{m^2(a^2-b^2)^2}{a^2+b^2m^2}$.

vi. The eccentric angle is measured on the auxiliary circle, not on the ellipse; ϕ is not the angle $\angle XCP$.

vii. A circle is the special case $a=b$, $e=0$, with the foci merged at the centre and the directrices at infinity.