[REVISION TOOL · JEE MAIN + ADVANCED · CLASS 11]

CONCEPT MAP

BINOMIAL THEOREM

5 TOPICS · 31 CORE IDEAS · ONE SHEET

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A THE THEOREM & ITS TERMS 01–07

01 THE BINOMIAL THEOREM A·01

For every positive integer n and all real or complex x and a,

KEY

$$(x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} a^r$$

written out,

$$(x+a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \cdots + {}^nC_n a^n$$

Putting $x = 1$ gives the form used most often:

$$(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$$

Alternating signs. Replacing a by $-a$ gives $(x-a)^n = \sum (-1)^r\, {}^nC_r x^{n-r} a^r$, so the terms alternate in sign starting with $+$.

02 FEATURES OF THE EXPANSION A·02

i. There are exactly $n+1$ terms.

ii. The sum of the exponents of x and a in every term is n.

iii. The exponent of x falls from n to 0 while the exponent of a rises from 0 to n.

iv. Coefficients equidistant from the two ends are equal, since ${}^nC_r = {}^nC_{n-r}$.

v. The coefficients ${}^nC_0, {}^nC_1, \ldots, {}^nC_n$ are the binomial coefficients, often written $C_0, C_1, \ldots, C_n$.

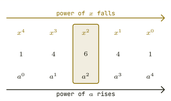
vi. The expansion is valid for every real or complex x and a when n is a positive integer; no convergence condition is needed.

03 GENERAL TERM A·03

The $(r+1)$th term of $(x+a)^n$ is

KEY

$$T_{r+1} = {}^nC_r\, x^{n-r} a^r$$



For $(x-a)^n$ the same term carries the sign $(-1)^r$:

$$T_{r+1} = (-1)^r\, {}^nC_r\, x^{n-r} a^r$$

Count from zero. T_{r+1} has r as its index, so the third term is T_3 with $r = 2$. Off-by-one here is the single commonest mistake in the chapter.

04 MIDDLE TERM A·04

The expansion of $(x+a)^n$ has $n+1$ terms, so:

KEY

n even : one middle term, $T_{\frac{n}{2}+1}$

and

n odd : two middle terms, $T_{\frac{n+1}{2}}$ and $T_{\frac{n+3}{2}}$

Middle term of $(1+x)^{2n}$. It is $T_{n+1} = {}^{2n}C_n x^n$, and ${}^{2n}C_n = \dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}\cdot 2^n$ — a form that turns up again and again.

05 TERMS FROM THE END A·05

The rth term from the end of $(x+a)^n$ is the $(n-r+2)$th term from the beginning:

KEY

$$T_r^{(\text{end})} = T_{n-r+2}$$

TERM INDEPENDENT OF x

Write down T_{r+1}, collect the whole power of x, set that exponent equal to zero and solve for r. If r is not a whole number between 0 and n, no such term exists.

Same method, three questions. The term independent of x, the coefficient of a given power, and the term containing a given power are all the same calculation with a different right-hand side.

06 PROPERTIES OF THE COEFFICIENTS A·06

i. ${}^nC_r = {}^nC_{n-r}$, so the coefficients read the same forwards and backwards.

ii. ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ — Pascal's rule.

iii. $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$, which is how one term is built from the last.

iv. $r\cdot {}^nC_r = n\cdot {}^{n-1}C_{r-1}$.

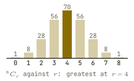
v. If ${}^nC_x = {}^nC_y$ with x, y whole numbers in $[0, n]$, then $x = y$ or $x + y = n$.

07 GREATEST COEFFICIENT A·07

Among ${}^nC_0, {}^nC_1, \ldots, {}^nC_n$ the greatest is

KEY

n even : ${}^nC_{n/2}$; $\quad n$ odd : ${}^nC_{\frac{n-1}{2}} = {}^nC_{\frac{n+1}{2}}$



Coefficient, not term. The greatest binomial coefficient is a property of n alone. The greatest **term**, by contrast, depends on x as well and is a different calculation — section 09.

B WORKING WITH THE EXPANSION 08–13

08 COEFFICIENT OF A GIVEN POWER B·08

To find the coefficient of x^m in an expansion, write the general term, gather the whole power of x, and equate that exponent to m.

KEY

$$T_{r+1} = {}^nC_r\, x^{n-r} \left(\frac{k}{x^p}\right)^r = {}^nC_r k^r x^{n-r-pr}$$

so the exponent is $n - r(1+p)$; set it equal to m and solve for r.

Reject a fractional r. If the resulting r is not a whole number between 0 and n, the expansion simply has no such term and the coefficient is zero.

09 NUMERICALLY GREATEST TERM B·09

For a given numerical x, compare consecutive terms of $(1+x)^n$:

KEY

$$\left|\frac{T_{r+1}}{T_r}\right| = \frac{n-r+1}{r}|x| \geq 1 \implies r \leq \frac{(n+1)|x|}{1+|x|}$$

i. Let $m = \dfrac{(n+1)|x|}{1+|x|}$.

ii. If m is not an integer, the greatest term is $T_{\lfloor m \rfloor + 1}$, where $\lfloor\ \rfloor$ is the greatest integer function.

iii. If m is an integer, the terms T_m and T_{m+1} are equal and both are greatest.

Reduce to $(1+x)^n$ first. Write $(p+q)^n = p^n\left(1+\frac{q}{p}\right)^n$ and apply the rule to the bracket; the factor p^n scales every term equally.

10 SUM AND DIFFERENCE OF TWO EXPANSIONS B·10

Adding and subtracting kills alternate terms:

KEY

$$(x+a)^n + (x-a)^n = 2\left[{}^nC_0 x^n + {}^nC_2 x^{n-2} a^2 + \cdots\right]$$

$$(x+a)^n - (x-a)^n = 2\left[{}^nC_1 x^{n-1} a + {}^nC_3 x^{n-3} a^3 + \cdots\right]$$

NUMBER OF TERMS

i. In the sum: $\dfrac{n+2}{2}$ terms if n is even, $\dfrac{n+1}{2}$ if n is odd.

ii. In the difference: $\dfrac{n}{2}$ terms if n is even, $\dfrac{n+1}{2}$ if n is odd.

Where it is used. Expressions like $(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6$ become rational at a stroke, because every surd sits in a term of odd index r — an even-numbered term — and those are exactly the terms that cancel.

11 NUMBER OF TERMS B·11

i. $(x+a)^n$: $n+1$ terms.

ii. $(x+y+z)^n$: $\dfrac{(n+1)(n+2)}{2}$ terms.

iii. $(x_1 + x_2 + \cdots + x_k)^n$: ${}^{n+k-1}C_{k-1}$ terms.

iv. $(1+x)^n(1-x)^n = (1-x^2)^n$: $n+1$ terms.

KEY

terms in $(x_1 + \cdots + x_k)^n = {}^{n+k-1}C_{k-1}$

Why that count. A term is fixed by the exponents $n_1 + n_2 + \cdots + n_k = n$ with each $n_i \geq 0$, and that equation has ${}^{n+k-1}C_{k-1}$ solutions.

12 MULTINOMIAL THEOREM B·12

For a positive integer n,

KEY

$$(x_1 + x_2 + \cdots + x_k)^n = \sum \frac{n!}{n_1! n_2! \cdots n_k!} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

the sum running over all non-negative integers with $n_1 + n_2 + \cdots + n_k = n$.

Reading a single coefficient. The coefficient of $x_1^{n_1} \cdots x_k^{n_k}$ is $\dfrac{n!}{n_1! \cdots n_k!}$ — the same number that counts the arrangements of n letters with those repetitions.

13 NEGATIVE AND FRACTIONAL INDEX B·13

For any real n, positive integer or not, and $|x| < 1$,

KEY

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

i. The series is infinite; it terminates only when n is a non-negative integer.

ii. The general term is $\dfrac{n(n-1)\cdots(n-r+1)}{r!}x^r$; the symbol ${}^nC_r$ is not used, because n need not be a whole number.

iii. The series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$; on $|x| = 1$ the answer depends on n. Treat $|x| < 1$ as compulsory.

C SUMS OF COEFFICIENTS 14–20

14 THE BASIC SUMS C·14

Putting $x = 1$ and $x = -1$ in $(1+x)^n = \sum C_r x^r$:

KEY

$$C_0 + C_1 + C_2 + \cdots + C_n = 2^n$$

$$C_0 - C_1 + C_2 - \cdots + (-1)^n C_n = 0 \quad (n \geq 1)$$

Adding and subtracting these two gives the odd and even splits:

$$C_0 + C_2 + C_4 + \cdots = C_1 + C_3 + C_5 + \cdots = 2^{n-1}$$

15 SUMS WITH R AS A FACTOR C·15

Differentiate $(1+x)^n = \sum C_r x^r$ and then substitute a value of x.

KEY

$$\sum_{r=0}^{n} r\, C_r = n\cdot 2^{n-1}$$

$$\sum_{r=0}^{n} r^2\, C_r = n(n+1)2^{n-2}$$

$$\sum_{r=0}^{n} (-1)^r\, r\, C_r = 0 \quad (n \geq 2)$$

The differentiation trick. Differentiating the identity and then multiplying by x — the operator $x\dfrac{d}{dx}$ — turns C_r into rC_r; applying it twice gives $r^2 C_r$. Choose $x = 1$ or $x = -1$ at the end.

16 SUMS WITH R IN THE DENOMINATOR C·16

Integrate $(1+x)^n = \sum C_r x^r$ from 0 to 1, or from -1 to 0.

KEY

$$\sum_{r=0}^{n} \frac{C_r}{r+1} = \frac{2^{n+1}-1}{n+1}$$

$$\sum_{r=0}^{n} \frac{(-1)^r C_r}{r+1} = \frac{1}{n+1}$$

The integration trick. Integration lowers C_r to $\dfrac{C_r}{r+1}$, exactly as differentiation raises it to rC_r. Pick the limits that produce the signs you want.

17 PRODUCTS OF TWO COEFFICIENTS C·17

Comparing coefficients on the two sides of $(1+x)^m(1+x)^n = (1+x)^{m+n}$ gives Vandermonde's identity:

KEY

$$\sum_{r=0}^{p} {}^mC_r\, {}^nC_{p-r} = {}^{m+n}C_p$$

The counting reading. Choosing p people from m men and n women, split by how many men are taken, is the same count as choosing p from $m+n$ at once. Read any ${}^mC_r$ with $r > m$ as zero, which is what keeps the limits simple.

18 SUMS OF SQUARES AND SHIFTED PRODUCTS C·18

Taking $m = n$ and $p = n$ in the last identity, and using $C_r = C_{n-r}$,

KEY

$$C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2 = {}^{2n}C_n$$

and more generally, for $0 \leq k \leq n$,

$$\sum_{r=0}^{n-k} C_r\, C_{r+k} = {}^{2n}C_{n-k}$$

Where it comes from. Replace x by $1/x$ in one factor: $\left(1+\frac{1}{x}\right)^n (1+x)^n = \dfrac{(1+x)^{2n}}{x^n}$. Comparing the coefficient of x^k on the two sides gives the identity.

19 THE REPLACEMENT METHOD C·19

Many sums are handled by writing the series twice, once forwards and once backwards, and adding. Using $C_r = C_{n-r}$,

KEY

$$\sum_{r=0}^{n} (a + rd)\, C_r = \left(a + \frac{nd}{2}\right) 2^n$$

i. Write $S = \sum_r f(r) C_r$ and also $S = \sum_r f(n-r) C_r$.

ii. Add the two; the coefficients pair up and $f(r) + f(n-r)$ is usually constant.

iii. Divide by two.

When it works. Whenever f is linear in r, the pairing makes $f(r) + f(n-r)$ independent of r and the whole sum collapses to a multiple of 2^n.

20 ROOTS OF UNITY FILTER C·20

To pick out the coefficients whose index is a multiple of 3, let ω be a non-real cube root of unity:

KEY

$$C_0 + C_3 + C_6 + \cdots = \frac{2^n + (1+\omega)^n + (1+\omega^2)^n}{3}$$

The same idea with i separates the indices modulo 4, and in general the kth roots of unity separate the indices modulo k.

Why it works. $1 + \omega^j + \omega^{2j}$ is 3 when j is a multiple of 3 and 0 otherwise, so averaging over the three roots keeps exactly the terms wanted.

D DIVISIBILITY & REMAINDERS 21–25

21 DIVISIBILITY D·21

Write the expression as a binomial expansion about a convenient base and read off the terms that survive.

KEY

$$(1+x)^n = 1 + nx + x^2\left[{}^nC_2 + {}^nC_3 x + \cdots\right]$$

so $(1+x)^n - 1 - nx$ is always divisible by x^2.

- $3^{2n+2} - 8n - 9$ is divisible by 64: write $3^{2n+2} = 9^{n+1} = (1+8)^{n+1}$.
- $3^n - 2n - 1$ is divisible by 4: write $3^n = (1+2)^n$.
- $a^n - b^n$ is divisible by $a - b$ for every positive integer n.

Choosing the base. Pick the base so that the divisor appears as the x in $(1+x)^n$; then every term from x^2 onwards carries the divisor at least twice.

22 REMAINDERS D·22

To find the remainder when N^k is divided by d, write N as a multiple of d plus a small number and expand:

KEY

$$N^k = (qd + s)^k = d\,[\cdots] + s^k$$

so the remainder is that of s^k, and the process is repeated until s^k is small.

Negative residues are quicker. Writing s as a negative number when it is close to d often halves the work: 18^k mod 19 is $(-1)^k$ mod 19.

23 LAST DIGITS D·23

i. For the last digit, work modulo 10; for the last two digits, modulo 100.

ii. Write the base as a multiple of 10 or 100 plus a small number and expand; only the last one or two terms survive.

iii. The last digits of powers repeat in a cycle, so reduce the exponent modulo the cycle length — but a remainder of 0 means the last member of the cycle, not the first.

KEY

$$(10k + a)^n \equiv a^n \pmod{10}$$

Two terms for two digits. Modulo 100 the expansion of $(10k+a)^n$ keeps $a^n + 10kn\,a^{n-1}$; everything beyond that carries a factor 100.

24 APPROXIMATION FOR SMALL X D·24

When $|x|$ is small enough that x^2 and higher powers may be dropped,

KEY

$$(1+x)^n \approx 1 + nx$$

and keeping one more term,

$$(1+x)^n \approx 1 + nx + \frac{n(n-1)}{2}x^2$$

How good is it? The error after the linear term is about $\dfrac{n(n-1)}{2}x^2$, so it grows with n: at $x = 0.01$ the linear form is good to four decimals only for very small n, and by $n = 10$ only to about two.

25 INTEGRAL AND FRACTIONAL PARTS D·25

Let $(p+\sqrt{q})^n = I + f$ where I is an integer and $0 < f < 1$, and let $f' = (p-\sqrt{q})^n$, where p, q are positive integers with $0 < p - \sqrt{q} < 1$.

KEY

$$(p+\sqrt{q})^n + (p-\sqrt{q})^n = \text{an integer}$$

because the odd powers of $\sqrt{q}$ cancel. So $f + f'$ is an integer strictly between 0 and 2, which forces $f + f' = 1$. Hence

$$(I + f)\, f' = (p+\sqrt{q})^n (p-\sqrt{q})^n = (p^2 - q)^n$$

The whole method in one line. Add the conjugate to make the surds cancel, then use $0 < f' < 1$ to pin down the fractional part exactly.

E SPECIAL SERIES & USES 26–31

26 STANDARD BINOMIAL SERIES E·26

For $|x| < 1$:

KEY

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$(1-x)^{-n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r\, x^r$$

- $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$
- $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$
- $(1+x)^{-n} = \sum (-1)^r\, {}^{n+r-1}C_r\, x^r$

27 EXPONENTIAL SERIES E·27

For every real x,

KEY

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

i. $e = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots$, and $2 < e < 3$.

ii. $e^{-1} = 1 - \dfrac{1}{1!} + \dfrac{1}{2!} - \cdots$, so $\dfrac{e + e^{-1}}{2} = \sum \dfrac{1}{(2n)!}$.

iii. $a^x = e^{x \log_e a} = 1 + \dfrac{x \log_e a}{1!} + \dfrac{(x \log_e a)^2}{2!} + \cdots$, valid for $a > 0$.

iv. $\sum_{n=1}^{\infty} \dfrac{n}{n!} = e$ and $\sum_{n=1}^{\infty} \dfrac{n^2}{n!} = 2e$.

28 LOGARITHMIC SERIES E·28

For $-1 < x \leq 1$,

KEY

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

and for $-1 \leq x < 1$,

$$\log_e(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)$$

Subtracting the two gives the fast-converging form

$$\log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

At $x = 1$. The first series gives $\log_e 2 = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \cdots$, the alternating harmonic series.

29 COUNTING WITH COEFFICIENTS E·29

The number of ways of choosing objects subject to limits is a coefficient in a product of binomial series.

KEY

$$\text{ways} = \text{coefficient of } x^r \text{ in } \prod_i (1 + x + \cdots + x^{a_i})$$

- Number of non-negative integer solutions of $x_1 + \cdots + x_k = n$: coefficient of x^n in $(1-x)^{-k}$, that is ${}^{n+k-1}C_{k-1}$.
- Number of ways of getting a total n on k dice: coefficient of x^n in $(x + x^2 + \cdots + x^6)^k$.

One idea, two chapters. This is the same coefficient method used in Permutations and Combinations; the binomial series is what makes the brackets easy to multiply.

30 STANDARD RESULTS WORTH MEMORISING E·30

i. $\sum C_r = 2^n$, $\sum (-1)^r C_r = 0$, $\sum_{\text{even}} C_r = \sum_{\text{odd}} C_r = 2^{n-1}$.

ii. $\sum r C_r = n 2^{n-1}$, $\sum r^2 C_r = n(n+1)2^{n-2}$.

iii. $\sum \dfrac{C_r}{r+1} = \dfrac{2^{n+1}-1}{n+1}$.

iv. $\sum C_r^2 = {}^{2n}C_n$.

v. ${}^{2n}C_n = \dfrac{2^n\,[1\cdot 3\cdot 5\cdots(2n-1)]}{n!}$.

vi. $C_1 + 2C_2 + 3C_3 + \cdots + nC_n = n 2^{n-1}$.

vii. $C_0 C_r + C_1 C_{r+1} + \cdots + C_{n-r} C_n = {}^{2n}C_{n-r}$.

31 QUICK FACTS AND COMMON TRAPS E·31

i. T_{r+1} carries the index r: the fifth term has $r = 4$, not $r = 5$.

ii. The greatest coefficient depends only on n; the greatest term depends on x as well.

iii. For a non-integral index there is no ${}^nC_r$ and no last term, and $|x| < 1$ is compulsory.

iv. $(x+a)^n$ and $(a+x)^n$ have the same value but their terms come in the opposite order.

v. Before using the greatest-term rule, reduce the bracket to the form $(1+x)^n$.

vi. In a two-term product, collect the whole power of x before equating exponents; forgetting a $1/x^p$ is the usual slip.

vii. Sums like $\sum rC_r$ come from differentiating, $\sum \dfrac{C_r}{r+1}$ from integrating; recognising which turns a long problem into two lines.