[REVISION TOOL · JEE MAIN + ADVANCED · CLASS 12]

CONCEPT MAP

DETERMINANTS

5 TOPICS · 28 CORE IDEAS · ONE SHEET

SCAN TO PRACTICE

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Ritesh Raj · M.Sc Mathematics, IIT Delhi
JEE Main · JEE Advanced · Olympiad Mathematics

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A VALUE OF A DETERMINANT 01-05

01 WHAT A DETERMINANT IS (A·01)

Every square matrix $A = [a_{ij}]$ of order n has a number, real or complex, attached to it. That number is the determinant of A, written $|A|$ or det A.

SECOND ORDER

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

THIRD ORDER

KEY

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

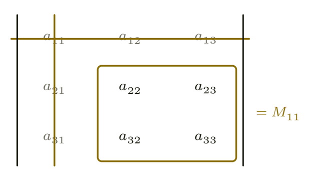
expanded along the first row, is

$$a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Matrix or determinant? A matrix is an array; a determinant is a single number computed from a square array. Only square matrices have one.

02 MINORS AND COFACTORS (A·02)

Let $A = (a_{ij})_{n\times n}$. The minor M_{ij} of the element a_{ij} is the determinant of the square submatrix of order $n-1$ obtained by deleting the ith row and the jth column of A.



The cofactor of a_{ij} is written C_{ij} and carries the chessboard sign:

KEY

$$C_{ij} = (-1)^{i+j} M_{ij}$$

The sign pattern. $(-1)^{i+j}$ is $+$ when $i+j$ is even. Laid out on the array it alternates like a chessboard, starting with $+$ at the top left.

03 EXPANSION AND THE COFACTOR SUM RULES (A·03)

A determinant may be expanded along any row or any column. Multiplying each element by its own cofactor gives the value of the determinant:

KEY

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

But multiplying the elements of one row by the cofactors of a different row gives zero:

$$a_{i1}C_{k1} + a_{i2}C_{k2} + \cdots + a_{in}C_{kn} = 0, \qquad i \neq k$$

Use it to save work. Expand along the row or column carrying the most zeros; each zero kills a whole cofactor and the arithmetic collapses.

04 TRANSPOSE, PRODUCT AND MULTIPLES (A·04)

For square matrices A and B of order n and a scalar k:

KEY

$$|A^T| = |A|, \qquad |AB| = |A|\,|B|, \qquad |kA| = k^n|A|$$

i. $|A^m| = |A|^m$ for $m \in N$.
ii. $|A^{-1}| = \frac{1}{|A|}$, provided $|A| \neq 0$.
iii. The determinant of the matrix of cofactors of A equals $|A|^{n-1}$.

Watch the k. $|kA| = k^n|A|$, not $k|A|$ — the scalar multiplies every one of the n rows, and property VII pulls one factor out of each.

05 SINGULAR AND NON-SINGULAR MATRICES (A·05)

A square matrix A is called singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

KEY

$$A^{-1} \text{ exists} \iff |A| \neq 0$$

- The product of two non-singular matrices of the same order is non-singular, since $|AB| = |A||B|$.
- If A is singular then so is AB, for every B.

One test, many uses. Invertibility, a unique solution of $AX = B$, full rank and linear independence of the rows are four statements of the single condition $|A| \neq 0$.

B PROPERTIES 06-10

06 THE FOUR BASIC PROPERTIES (B·06)

i. The value of a determinant is unchanged when its rows are written as columns and its columns as rows. Thus $|A| = |A^T|$.
ii. If any two rows, or any two columns, are interchanged, the determinant changes sign.
iii. If two rows, or two columns, are identical, the determinant is zero.
iv. If two rows, or two columns, are proportional, the determinant is zero.

Three and four follow from two. Swapping the two identical rows leaves the determinant unchanged and, by property II, also negates it; the only number equal to its own negative is zero.

07 ZERO ROWS, TRIANGLES AND MULTIPLES (B·07)

i. If every element of a row, or of a column, is zero, the determinant is zero.
ii. If all the elements above the main diagonal are zero, or all those below it are, the determinant is the product of the diagonal elements.
iii. If every element of one row, or of one column, is multiplied by a constant k, the new determinant is k times the original.

KEY

$$\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Read it both ways. The same property lets you take a common factor out of a row, which is usually the first move in simplifying a determinant by hand.

08 SPLITTING AND ROW OPERATIONS (B·08)

SPLITTING A ROW

If every element of a row, or of a column, is the sum of two or more terms, the determinant splits into the sum of two or more determinants:

KEY

$$\begin{vmatrix} a_1+p_1 & b_1 & c_1 \\ a_2+p_2 & b_2 & c_2 \\ a_3+p_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} p_1 & b_1 & c_1 \\ p_2 & b_2 & c_2 \\ p_3 & b_3 & c_3 \end{vmatrix}$$

ROW OPERATIONS

If the elements of one row, or column, are multiplied by a constant k and added to the corresponding elements of another row, or column, the value of the determinant is unchanged:

$$R_i \to R_i + kR_j \ (i \neq j) \ \text{ leaves } |A| \text{ unchanged}$$

Only one at a time. The operation is safe when the row being changed is not the row being used. Chaining two operations that each modify a row the other one uses will change the value.

09 THE FACTOR THEOREM FOR DETERMINANTS (B·09)

Suppose the elements of a determinant are polynomials in x. If the determinant vanishes at $x = a$, then $(x-a)$ is a factor of it.

The usual way this happens is that two rows, or two columns, become identical when a is put in place of x.

REPEATED FACTORS

More generally, if r rows or r columns become identical when a is substituted for x, then

KEY

$$(x-a)^{r-1} \text{ is a factor of the determinant}$$

Degree check. Count the highest power of x the expansion can produce, then match it against the factors you have found; the leftover is a constant you can fix from a single convenient value of x.

10 PRODUCT OF TWO DETERMINANTS (B·10)

Two determinants of the same order are multiplied row by row: the (i,j) entry of the product is the sum of the products of the ith row of the first with the jth row of the second.

KEY

$$\Delta_1\Delta_2 = \begin{vmatrix} \Sigma a_1\alpha_1 & \Sigma a_1\alpha_2 & \Sigma a_1\alpha_3 \\ \Sigma a_2\alpha_1 & \Sigma a_2\alpha_2 & \Sigma a_2\alpha_3 \\ \Sigma a_3\alpha_1 & \Sigma a_3\alpha_2 & \Sigma a_3\alpha_3 \end{vmatrix}$$

where, for instance, $\Sigma a_1\alpha_2 = a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2$, taking Δ_1 with rows (a_i, b_i, c_i) and Δ_2 with rows $(\alpha_j, \beta_j, \gamma_j)$.

Any of four ways. Because $|A| = |A^T|$, the multiplication may be done row by row, row by column, column by row or column by column — whichever makes the entries simplest.

C STANDARD FORMS & CALCULUS 11-16

11 STANDARD DETERMINANTS WORTH KNOWING (C·11)

VANDERMONDE

KEY

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

THE CYCLIC DETERMINANT

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc)$$

and, in factored form, this equals $-(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$.

ONE MORE

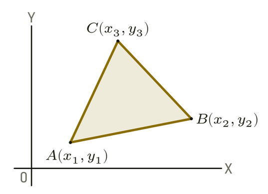
$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

12 AREA OF A TRIANGLE AND COLLINEARITY (C·12)

For the triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3),

KEY

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$



The determinant is a signed quantity, so take its modulus for the area. The three points are collinear exactly when it vanishes:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

13 EQUATION OF A LINE AS A DETERMINANT (C·13)

A point (x, y) lies on the line through (x_1, y_1) and (x_2, y_2) exactly when the three points are collinear. Writing that as a determinant gives the equation of the line at once:

KEY

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Why it is worth the ink. No slope appears, so the form survives the vertical case $x_1 = x_2$ that breaks $y - y_1 = m(x - x_1)$.

14 DIFFERENTIATION OF DETERMINANTS (C·14)

Let the entries be functions of x:

KEY

$$\Delta(x) = \begin{vmatrix} f_1(x) & g_1(x) \\ f_2(x) & g_2(x) \end{vmatrix}$$

Differentiating row by row,

$$\Delta'(x) = \begin{vmatrix} f_1'(x) & g_1'(x) \\ f_2(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) \\ f_2'(x) & g_2'(x) \end{vmatrix}$$

and, equally, column by column,

$$\Delta'(x) = \begin{vmatrix} f_1'(x) & g_1(x) \\ f_2'(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1'(x) \\ f_2(x) & g_2'(x) \end{vmatrix}$$

The rule in words. Differentiate one row, or one column, at a time and leave the others alone, then add the results. For an $n \times n$ determinant this gives n terms.

15 INTEGRATION OF DETERMINANTS (C·15)

If only one row, or one column, contains functions of x and the rest are constants,

KEY

$$\Delta(x) = \begin{vmatrix} f(x) & g(x) \\ \lambda_1 & \lambda_2 \end{vmatrix}$$

then the integral may be taken inside that row:

$$\int_a^b \Delta(x)\,dx = \begin{vmatrix} \int_a^b f(x)\,dx & \int_a^b g(x)\,dx \\ \lambda_1 & \lambda_2 \end{vmatrix}$$

Only one variable row. Here λ_1 and λ_2 must be constants. If two rows contain functions of x, the determinant has to be expanded before integrating.

16 GREATEST AND LEAST VALUE OF A DETERMINANT (C·16)

Suppose the nine entries of

KEY

$$|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$$

are all drawn from a given set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Then $|A|$ is greatest when the diagonal entries are all the smallest member of the set and the off-diagonal entries are all the largest:

$$|A|_{\text{max}} \text{ at } a_{ii} = \min(\alpha_k), \quad a_{ij} = \max(\alpha_k)$$

and the least value is the negative of the greatest:

$$|A|_{\text{min}} = -|A|_{\text{max}}$$

D ADJOINT, INVERSE & RANK 17-22

17 ADJOINT OF A SQUARE MATRIX (D·17)

Let $A = [a_{ij}]$ be square of order n and let C_{ij} be the cofactor of a_{ij}. The adjoint of A, written adj(A), is the transpose of the matrix of cofactors.

KEY

$$[\text{adj}(A)]_{ij} = C_{ji}$$

In words: put in the (i,j) place of adj(A) the cofactor of the (j,i) element of A.

Do not forget the transpose. Building the cofactor matrix and stopping there is the commonest error in this topic; the adjoint is that matrix turned about its main diagonal.

18 PROPERTIES OF THE ADJOINT (D·18)

Let A be a square matrix of order n.

KEY

$$A\,(\text{adj}\,A) = |A|\,I_n = (\text{adj}\,A)\,A$$

i. $|\text{adj}\,A| = |A|^{n-1}$, if $|A| \neq 0$.
ii. adj(adj A) $= |A|^{n-2}A$, if $|A| \neq 0$.
iii. $|\text{adj}(\text{adj}\,A)| = |A|^{(n-2)^2}$, if $|A| \neq 0$.
iv. adj$(A^T) = (\text{adj}\,A)^T$.
v. adj$(A^m) = (\text{adj}\,A)^m$, $m \in N$.
vi. adj$(kA) = k^{n-1}(\text{adj}\,A)$, $k \in R$.
vii. adj$(AB) = (\text{adj}\,B)(\text{adj}\,A)$.

The first identity is the engine. Dividing $A(\text{adj}\,A) = |A|I$ by $|A|$ is exactly where the formula for the inverse comes from.

19 INVERSE OF A SQUARE MATRIX (D·19)

The inverse of a square matrix A, written A^{-1}, is given by

KEY

$$A^{-1} = \frac{1}{|A|}\,\text{adj}(A), \qquad |A| \neq 0$$

It is the unique matrix with $AA^{-1} = A^{-1}A = I$. A singular matrix has no inverse.

Order of work. Compute $|A|$ first. If it is zero you can stop; if not, build the cofactors, transpose, and divide.

20 PROPERTIES OF THE INVERSE (D·20)

i. $(A^{-1})^{-1} = A$.
ii. $AA^{-1} = A^{-1}A = I$.
iii. $(A^T)^{-1} = (A^{-1})^T$.
iv. $(AB)^{-1} = B^{-1}A^{-1}$, and $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.
v. adj$(A^{-1}) = (\text{adj}\,A)^{-1}$.
vi. If $A = \text{diag}(a_1, a_2, \ldots, a_n)$ then $A^{-1} = \text{diag}(a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1})$.
vii. $|A^{-1}| = |A|^{-1}$.

The reversal law. Inverting a product reverses the order of the factors, exactly as taking off shoes and socks reverses the order in which they were put on.

21 RANK OF A MATRIX (D·21)

A positive integer r is the rank of a non-zero matrix A if

i. there is at least one minor of A of order r that is not zero, and
ii. every minor of A of order greater than r is zero.

It is written $\rho(A) = r$.

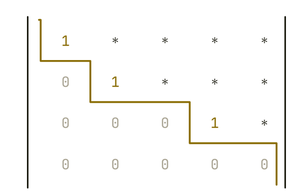
PROPERTIES

i. The rank of a zero (null) matrix is zero: $\rho(O) = 0$.
ii. $\rho(I_n) = n$ for the unit matrix of order n.
iii. If A is an $n \times n$ non-singular matrix, $\rho(A) = n$.
iv. Elementary transformations do not change the rank of a matrix.

22 ECHELON FORM (D·22)

A matrix A is in Echelon form if it is the null matrix, or else satisfies all of:

i. every non-zero row comes before every zero row;
ii. the number of zeros before the first non-zero element of a row is less than the number of such zeros in the row beneath it;
iii. the first non-zero element in a row is unity.



Why it is useful. The number of non-zero rows of a matrix written in Echelon form is its rank — which turns a search through every minor into a few row operations.

E SYSTEMS OF LINEAR EQUATIONS 23-28

23 MATRIX METHOD: NON-HOMOGENEOUS SYSTEM (E·23)

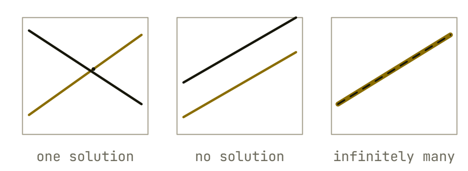
A system of n linear equations in n variables is written $AX = B$, where A is the coefficient matrix, X the column of unknowns and B the column of constants.

KEY

$$AX = B$$

THE THREE CASES

i. If $|A| \neq 0$, the system is consistent with the unique solution $X = A^{-1}B$.
ii. If $|A| = 0$ and $(\text{adj}\,A)B \neq O$, the system is inconsistent and has no solution.
iii. If $|A| = 0$ and $(\text{adj}\,A)B = O$, the system may be consistent or inconsistent; if it is consistent it has infinitely many solutions.



24 MATRIX METHOD: HOMOGENEOUS SYSTEM (E·24)

The system $AX = B$ is homogeneous when B is the zero matrix, so it takes the form

KEY

$$AX = O$$

i. If $|A| \neq 0$, the only solution is $X = O$, called the trivial or zero solution.
ii. If $|A| = 0$, the system has non-trivial solutions, and there are infinitely many of them.

Never inconsistent. $X = O$ always satisfies a homogeneous system, so the only question is whether anything else does.

25 CRAMER'S RULE (E·25)

For the system $a_1x+b_1y+c_1z = d_1$, $a_2x+b_2y+c_2z = d_2$, $a_3x+b_3y+c_3z = d_3$, put

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \qquad D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

so that D_k is D with its kth column replaced by the column of constants. Then

KEY

$$x = \frac{D_1}{D}, \qquad y = \frac{D_2}{D}, \qquad z = \frac{D_3}{D}$$

26 CRAMER'S RULE: READING THE CASES (E·26)

NON-HOMOGENEOUS SYSTEM

i. If $D \neq 0$, the system is consistent with the unique solution $x = D_1/D$, $y = D_2/D$, $z = D_3/D$.
ii. If $D = 0$ and at least one of D_1, D_2, D_3 is non-zero, the system is inconsistent and has no solution.
iii. If $D = D_1 = D_2 = D_3 = 0$, the system may or may not be consistent; if it is consistent it has infinitely many solutions.

HOMOGENEOUS SYSTEM

Here $D_1 = D_2 = D_3 = 0$ automatically, since each of those determinants has a column of zeros. So:

i. If $D \neq 0$, only the trivial solution $x = y = z = 0$.
ii. If $D = 0$, infinitely many solutions.

27 RANK METHOD (E·27)

Let $AX = B$ be a system of n linear equations in n variables.

i. Write the augmented matrix $[A : B]$.
ii. Reduce it to Echelon form by elementary row, or column, operations.
iii. Read off $\rho(A)$ and $\rho[A : B]$ by counting non-zero rows in the Echelon form of A and of $[A : B]$.
iv. Compare the two ranks against the number of unknowns.

RANKS	CONCLUSION
$\rho(A) \neq \rho[A{:}B]$	inconsistent, no solution
$\rho(A) = \rho[A{:}B] =$ unknowns	consistent, unique solution
$\rho(A) = \rho[A{:}B] <$ unknowns	consistent, infinitely many

HOMOGENEOUS SYSTEM $AX = O$

i. If $\rho(A) =$ number of variables, only the trivial solution $X = O$.
ii. If $\rho(A) <$ number of variables, infinitely many non-trivial solutions.

28 CHARACTERISTIC EQUATION AND EIGENVALUES (E·28)

For $A = [a_{ij}]_{n\times n}$, the equation

KEY

$$|A - \lambda I_n| = 0$$

is called the characteristic equation of A, and its roots λ are the eigenvalues of A.

PROPERTIES

i. The sum of the eigenvalues of A equals its trace.
ii. The product of the eigenvalues of A equals $|A|$.
iii. The eigenvalues of a triangular matrix are its diagonal elements.
iv. If λ is an eigenvalue of A and $|A| \neq 0$, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} and $\frac{|A|}{\lambda}$ is an eigenvalue of adj(A).
v. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, then $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$ are the eigenvalues of A^2.