CONCEPT MAP

HYPERBOLA

5 TOPICS · 31 CORE IDEAS · ONE SHEET

SCAN TO PRACTICE

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A THE CURVE & ITS ELEMENTS 01-07

01 DEFINITION AND STANDARD EQUATION A·01

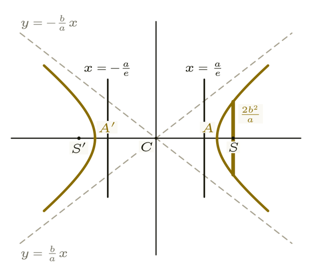
A hyperbola is the locus of a point that moves so that its distance from a fixed point S (the focus) is a constant multiple $e>1$ of its distance from a fixed line (the directrix):

KEY

$$SP = e\cdot PM, \qquad e>1$$

Taking the focus at $(ae, 0)$ and the directrix $x = a/e$ gives the standard equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \qquad b^2=a^2(e^2-1)$$



One number decides the conic. $e<1$ gives an ellipse, $e=1$ a parabola and $e>1$ a hyperbola. Everything that looks different about the hyperbola follows from $e>1$, which makes $b^2=a^2(e^2-1)$ positive and puts a minus sign in the equation.

02 THE ELEMENTS OF THE CURVE A·02

ELEMENT	FOR $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$
Centre	$(0,0)$
Vertices	$(a,0)$ and $(-a,0)$
Foci	$(\pm ae, 0)$
Transverse axis	$y=0$, length $2a$
Conjugate axis	$x=0$, length $2b$
Directrices	$x=\pm a/e$
Latus rectum	$\frac{2b^2}{a}=2a(e^2-1)$
Ends of a latus rectum	$\left(\pm ae,\ \pm\frac{b^2}{a}\right)$
Distance between foci	$2ae$
Distance between directrices	$2a/e$

The curve never crosses the conjugate axis. Putting $x=0$ gives $y^2=-b^2$, so b is a length that governs the shape without ever being reached. The transverse axis carries the vertices; the conjugate axis carries nothing.

03 ECCENTRICITY A·03

KEY

$$e^2 = 1+\frac{b^2}{a^2} = 1+\frac{(\text{conjugate axis})^2}{(\text{transverse axis})^2}$$

equivalently $b^2=a^2(e^2-1)$, and

$$e=\sqrt{\frac{a^2+b^2}{a^2}} > 1 \text{ always}$$

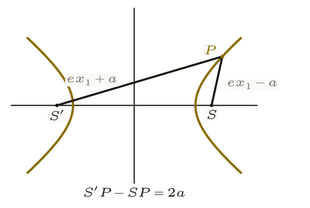
Sign is the whole story. The ellipse has $b^2=a^2(1-e^2)$ and the hyperbola $b^2=a^2(e^2-1)$. Copying the ellipse formula here is the commonest slip in the chapter.

04 FOCAL DISTANCES A·04

For $P(x_1, y_1)$ on the branch nearer $S(ae, 0)$,

KEY

$$SP = ex_1 - a, \qquad S'P = ex_1 + a$$



so for every point of the curve

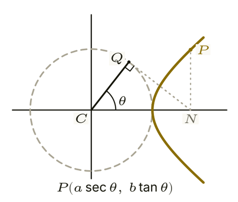
$$|S'P-SP|=2a$$

Difference, not sum. The ellipse keeps $S'P+SP$ constant; the hyperbola keeps the difference constant. The constant is the transverse axis in both cases.

05 PARAMETRIC FORM AND THE AUXILIARY CIRCLE A·05

KEY

$$x=a\sec\theta, \qquad y=b\tan\theta$$



i. θ runs over $[0, 2\pi)$ with $\theta\ne\pi/2,\ 3\pi/2$; the excluded values are where $\sec\theta$ blows up.
ii. $\theta\in(-\pi/2, \pi/2)$ traces the right branch and $\theta\in(\pi/2, 3\pi/2)$ the left.
iii. The circle $x^2+y^2=a^2$ on the transverse axis as diameter is the auxiliary circle. If N is the foot of the ordinate of P and NQ touches the circle at Q, then θ is the angle QCN.
iv. $x=a\cosh t,\ y=b\sinh t$ is an alternative that covers the right branch alone.

θ is not the angle PCN. For the ellipse the eccentric angle is read straight off the auxiliary circle; here it is read off the tangent from N, so the picture is different even though the name is the same.

06 THE CONJUGATE HYPERBOLA A·06

Swapping the roles of the two axes gives the conjugate hyperbola $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

	HYPERBOLA	CONJUGATE
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Foci	$(\pm ae, 0)$	$(0, \pm be')$
Transverse axis	$2a$	$2b$
Conjugate axis	$2b$	$2a$
Directrices	$x=\pm a/e$	$y=\pm b/e'$
Eccentricity	$e^2=\frac{a^2+b^2}{a^2}$	$e'^2=\frac{a^2+b^2}{b^2}$
Latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$

KEY

$$\frac{1}{e^2}+\frac{1}{e'^2}=1$$

07 OTHER FORMS AND A SHIFTED CENTRE A·07

With the transverse axis along OY:

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1: \text{ foci } (0, \pm ae), \text{ vertices } (0, \pm a)$$

Shifting the centre to (h, k) leaves every formula intact once x and y are replaced by $x-h$ and $y-k$:

KEY

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

IS A GENERAL CONIC A HYPERBOLA?

$ax^2+2hxy+by^2+2gx+2fy+c=0$ is a hyperbola when $\Delta\ne 0$ and $h^2>ab$, and a rectangular hyperbola when in addition $a+b=0$.

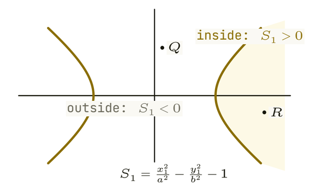
B POINT, LINE & CHORD 08-13

08 POSITION OF A POINT B·08

Write S_1 for the left-hand side evaluated at the point:

KEY

$$S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$$



- $S_1>0$: the point lies **inside** a branch, on the same side as a focus.
- $S_1=0$: the point lies on the curve.
- $S_1<0$: the point lies **outside**, in the region between the branches.

Correction to the printed sheet. The source has these three cases the wrong way round. Test it on the focus: $S_1=e^2-1>0$ there, and no real tangent can be drawn from a focus, so $S_1>0$ has to be the inside. The source's own rule for the number of tangents (section 13) agrees with the version above.

09 POSITION OF A LINE B·09

Substituting $y=mx+c$ leaves a quadratic in x whose discriminant has the sign of $c^2-(a^2m^2-b^2)$. The line cuts the hyperbola in two points, touches it, or misses it according as

$$c^2>a^2m^2-b^2, \quad =, \quad <$$

so the condition for tangency is

KEY

$$c^2=a^2m^2-b^2$$

Not every slope has a tangent. Tangency needs $a^2m^2>b^2$, that is $|m|>b/a$. Lines flatter than an asymptote never touch the curve, and a line exactly parallel to an asymptote meets it in one point without being a tangent.

10 CHORD JOINING TWO POINTS B·10

The chord through the points with parameters θ_1 and θ_2 is

KEY

$$\frac{x}{a}\cos\frac{\theta_1-\theta_2}{2}-\frac{y}{b}\sin\frac{\theta_1+\theta_2}{2}=\cos\frac{\theta_1+\theta_2}{2}$$

FOCAL CHORD

Putting $(ae, 0)$ in that equation gives the condition for the chord to pass through the focus:

$$e=\frac{\cos\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1-\theta_2}{2}}$$

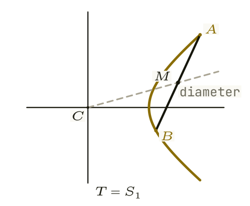
Set $\theta_2=\theta_1$ to recover the tangent. Every chord formula in this chapter collapses to the tangent at θ_1 when the two parameters are made equal — a free check on whether you have written it down correctly.

11 CHORD WITH A GIVEN MID-POINT B·11

The chord of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ whose mid-point is (x_1, y_1) is $T=S_1$, that is

KEY

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}$$



so its slope is

$$m=\frac{b^2x_1}{a^2y_1}$$

T and S_1. T is the tangent expression with $x^2\to xx_1$, $y^2\to yy_1$; S_1 is the curve's own expression at (x_1, y_1). Tangent is $T=0$, chord of contact is $T=0$, mid-point chord is $T=S_1$, pair of tangents is $SS_1=T^2$.

12 DIAMETERS AND CONJUGATE DIAMETERS B·12

The locus of the mid-points of a family of parallel chords of slope m is a diameter:

KEY

$$y=\frac{b^2}{a^2m}x$$

Two diameters $y=m_1x$ and $y=m_2x$ are conjugate when each bisects the chords parallel to the other:

$$m_1m_2=\frac{b^2}{a^2}$$

Only one of a conjugate pair meets the curve. Since $m_1m_2=b^2/a^2>0$ the two slopes have the same sign, and one is steeper than b/a while the other is flatter. The **steeper** diameter misses the hyperbola and meets the conjugate hyperbola instead; the flatter one carries the real chords. Check it on $m=0$, the transverse axis, which meets the curve at the vertices.

13 PAIR OF TANGENTS FROM A POINT B·13

The two tangents from $P(x_1, y_1)$ together have the combined equation

KEY

$$SS_1=T^2$$

where $S=\frac{x^2}{a^2}-\frac{y^2}{b^2}-1$, S_1 is S at (x_1, y_1) and $T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$.

- $S_1<0$ (outside): two real distinct tangents.
- $S_1=0$ (on the curve): the two coincide.
- $S_1>0$ (inside): no real tangent.

One exception. From a point on an asymptote the quadratic still has two roots, but one of them reproduces the asymptote, which touches the curve nowhere. Such a point has only one genuine tangent.

C TANGENTS 14-20

14 TANGENT IN SLOPE FORM C·14

KEY

$$y=mx\pm\sqrt{a^2m^2-b^2}$$

valid for every m with $a^2m^2>b^2$. The points of contact are

$$\left(\mp\frac{a^2m}{\sqrt{a^2m^2-b^2}},\ \mp\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$$

Two tangents per slope. The $\pm$ gives one tangent to each branch, and they are parallel. The signs are crossed: the upper sign in the tangent goes with the **lower** sign in the point of contact. In the normal of section 22 they are not crossed.

15 TANGENT AT A POINT C·15

At (x_1, y_1) on the curve:

KEY

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$$

and at the parametric point $(a\sec\theta,\ b\tan\theta)$:

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$$

Read it off the curve. Replace x^2 by xx_1 and y^2 by yy_1 in the equation of the hyperbola. The same substitution gives the tangent to any conic written in standard form.

16 POINT OF INTERSECTION OF TWO TANGENTS C·16

The tangents at θ_1 and θ_2 meet at

KEY

$$\left(a\frac{\cos\frac{\theta_1-\theta_2}{2}}{\cos\frac{\theta_1+\theta_2}{2}},\ b\tan\frac{\theta_1+\theta_2}{2}\right)$$

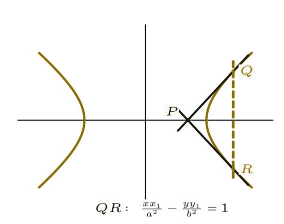
Where it is used. Any locus question of the form "two tangents meet at right angles" or "$\theta_1+\theta_2$ is constant" is answered by eliminating the parameters from these two coordinates.

17 CHORD OF CONTACT C·17

If the tangents from $P(x_1, y_1)$ touch the hyperbola at Q and R, then QR is

KEY

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$$



Same expression, three jobs. $T=0$ is the tangent when (x_1, y_1) is on the curve, the chord of contact when it is outside, and the polar in every case.

18 POLE AND POLAR C·18

The polar of $P(x_1, y_1)$ with respect to the hyperbola is $T=0$:

KEY

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$$

and the pole of the line $lx+my+n=0$ is

$$\left(-\frac{a^2l}{n},\ \frac{b^2m}{n}\right)$$

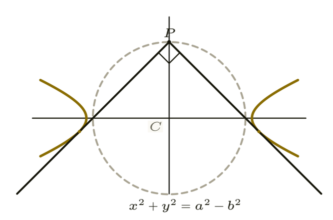
Conjugate points. If the polar of P passes through Q, then the polar of Q passes through P. Such a pair is called conjugate, and the relation is symmetric because T is symmetric in the two sets of coordinates.

19 DIRECTOR CIRCLE C·19

The locus of the point from which two perpendicular tangents can be drawn is

KEY

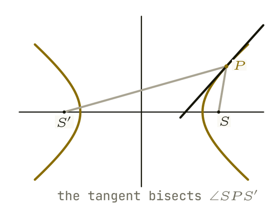
$$x^2+y^2=a^2-b^2$$



- $a>b$: a real circle of radius $\sqrt{a^2-b^2}$.
- $a=b$ (rectangular): the circle shrinks to the centre, where the two perpendicular asymptotes meet.
- $a<b$: no real locus — no point of the plane has two perpendicular tangents.

20 REFLECTION PROPERTY C·20

The tangent at P bisects the angle SPS' between the two focal radii, so the normal bisects the exterior angle.



A ray aimed at the far focus is therefore reflected by the near branch straight into the near focus — the property a Cassegrain telescope is built on. Two further facts about the foot Y of the perpendicular from S to a tangent:

KEY

$$Y \text{ lies on } x^2+y^2=a^2, \qquad SY\cdot S'Y'=b^2$$

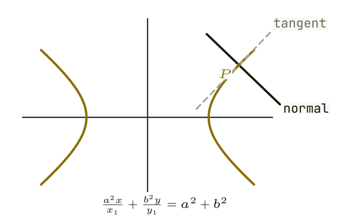
Opposite to the ellipse. On an ellipse the tangent bisects the **exterior** angle between the focal radii, which is why light from one focus converges on the other. On the hyperbola it is the interior angle, so the light diverges.

D NORMALS & ASYMPTOTES 21-26

21 NORMAL AT A POINT D·21

KEY

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$$



In parametric form, at $(a\sec\theta,\ b\tan\theta)$,

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$$

Plus, not minus. The tangent carries a minus sign between its two terms and the normal a plus. The right-hand side is a^2+b^2 for the hyperbola and a^2-b^2 for the ellipse.

22 NORMAL IN SLOPE FORM D·22

The normal of slope m is

KEY

$$y=mx\mp\frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$$

with points of contact

$$\left(\pm\frac{a^2}{\sqrt{a^2-b^2m^2}},\ \mp\frac{mb^2}{\sqrt{a^2-b^2m^2}}\right)$$

Real only when $a^2>b^2m^2$. Steep normals do not exist, which is the mirror image of the restriction $|m|>b/a$ on tangents.

23 CONORMAL POINTS D·23

Four normals can in general be drawn from a point; their feet are the conormal points. If $\theta_1, \theta_2, \theta_3, \theta_4$ are their eccentric angles, then $\theta_1+\theta_2+\theta_3+\theta_4$ is an odd multiple of π, and

$$\sum\cos(\theta_1+\theta_2)=0, \qquad \sum\sin(\theta_1+\theta_2)=0$$

If the normals at P, Q, R, S are concurrent then

KEY

$$(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4$$

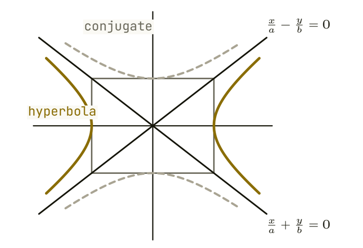
Three-point version. If $\theta_1, \theta_2, \theta_3$ satisfy $\sin(\theta_1+\theta_2)+\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_1)=0$, the normals at those three points are concurrent.

24 ASYMPTOTES D·24

An asymptote is a line that the tangent approaches as the point of contact runs off to infinity. For the standard hyperbola they are

KEY

$$y=\pm\frac{b}{a}x \quad\text{or}\quad \frac{x}{a}\pm\frac{y}{b}=0$$



Their combined equation is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$, and they are the diagonals of the rectangle $x=\pm a,\ y=\pm b$.

How to spot them. Set the constant term of the standard equation to zero. For a general conic, the asymptotes are $S+\lambda=0$ with λ chosen to make the equation represent a pair of lines.

25 HYPERBOLA, ASYMPTOTES AND CONJUGATE D·25

The three curves differ only in the constant term:

KEY

$$H+H'=2A$$

where $H: \frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$, $A: \frac{x^2}{a^2}-\frac{y^2}{b^2}=0$ and $H': \frac{x^2}{a^2}-\frac{y^2}{b^2}+1=0$.

- A hyperbola and its conjugate have the same asymptotes and the same centre.
- The asymptotes bisect the angles between the transverse and conjugate axes only when $a=b$.
- Any line through the centre other than an asymptote meets exactly one of the two conjugate hyperbolas.

26 ANGLE BETWEEN THE ASYMPTOTES D·26

KEY

$$\phi=2\tan^{-1}\frac{b}{a}=2\sec^{-1}e$$

- $\phi=\pi/2$ exactly when $a=b$, that is when $e=\sqrt{2}$.
- The transverse axis bisects the angle in which the curve lies.
- The perpendicular distance from either focus to either asymptote is b, and the foot of that perpendicular is at distance a from the centre.

A quick route to e. If a question gives the angle between the asymptotes, use $e=\sec(\phi/2)$ and skip a and b entirely.

E RECTANGULAR HYPERBOLA 27-31

27 RECTANGULAR HYPERBOLA E·27

When the transverse and conjugate axes are equal, $a=b$ and the equation becomes

KEY

$$x^2-y^2=a^2, \qquad e=\sqrt{2}$$

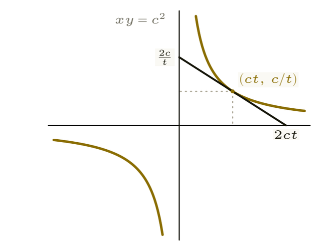
- The asymptotes are $y=\pm x$, which are perpendicular — hence the name.
- The latus rectum $2b^2/a=2a$ equals the transverse axis.
- Its conjugate $y^2-x^2=a^2$ is again rectangular with the same asymptotes.

28 THE FORM $xy=c^2$ E·28

Turning the axes through $45°$ so that they lie along the asymptotes turns $x^2-y^2=a^2$ into

KEY

$$xy=c^2, \qquad a=c\sqrt{2}$$



ELEMENT	FOR $xy=c^2$
Parametric point	$(ct,\ c/t)$, $t\ne 0$
Centre	$(0,0)$
Vertices	(c,c) and $(-c,-c)$
Foci	$(c\sqrt{2}, c\sqrt{2})$ and $(-c\sqrt{2}, -c\sqrt{2})$
Directrices	$x+y=\pm c\sqrt{2}$
Latus rectum	$2\sqrt{2}\,c$
Eccentricity	$\sqrt{2}$

29 TANGENT AND NORMAL TO $xy=c^2$ E·29

At (x_1, y_1) the tangent is

KEY

$$xy_1+yx_1=2c^2 \quad\text{or}\quad \frac{x}{x_1}+\frac{y}{y_1}=2$$

and at $(ct,\ c/t)$,

$$\frac{x}{t}+yt=2c$$

The normal at (x_1, y_1) is $xx_1-yy_1=x_1^2-y_1^2$, and at $(ct,\ c/t)$,

$$xt^3-yt-ct^4+c=0$$

Where the normal cuts again. If the normal at t_1 meets the curve again at t_2, then $t_1^3t_2=-1$.

30 PROPERTIES OF $xy=c^2$ E·30

The tangent at $(ct,\ c/t)$ meets the axes at $(2ct, 0)$ and $(0, 2c/t)$, so

KEY

$$\text{area of the triangle it cuts off} = 2c^2$$

i. The point of contact is the mid-point of the piece of the tangent between the asymptotes.
ii. The chord joining t_1 and t_2 is $x+t_1t_2\,y=c(t_1+t_2)$.
iii. The tangents at t_1 and t_2 meet at $\left(\frac{2ct_1t_2}{t_1+t_2},\ \frac{2c}{t_1+t_2}\right)$ — the harmonic means of the coordinates.
iv. The chord of contact from (x_1, y_1) is $xy_1+yx_1=2c^2$.
v. Four points t_1, t_2, t_3, t_4 are concyclic if and only if $t_1t_2t_3t_4=1$.

31 QUICK FACTS AND COMMON TRAPS E·31

i. $b^2=a^2(e^2-1)$ for the hyperbola, $a^2(1-e^2)$ for the ellipse. Get the sign right first.
ii. Tangency is $c^2=a^2m^2-b^2$, not $a^2m^2+b^2$; and it needs $|m|>b/a$.
iii. $S_1>0$ means inside, $S_1<0$ outside — the reverse of the ellipse.
iv. A line parallel to an asymptote, but not the asymptote itself, meets the hyperbola in exactly one point and is not a tangent.
v. The director circle is imaginary when $b>a$, so "perpendicular tangents" questions can have no answer.
vi. a need not exceed b; the transverse axis is decided by which term is positive, not by which letter is larger.
vii. For $xy=c^2$ the asymptotes are the axes, so the centre is the origin and the vertices lie on $y=x$.