CONCEPT MAP

SOLUTION OF TRIANGLE

5 TOPICS · 19 CORE IDEAS · ONE SHEET

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SCAN TO PRACTICE

A THE THREE RULES 01–05

01 SINE RULE
A · 01

In any triangle the sides are proportional to the sines of the angles opposite them:

KEY

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

where R is the radius of the circumcircle of the triangle.

Reading it. The common ratio is not just a constant, it is $2R$. That single fact links every angle of the triangle to its circumradius and is what makes the sine rule so useful in circle problems.

02 COSINE RULE
A · 02

If the three sides are known, every angle follows from

KEY

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \qquad \cos B = \frac{c^2+a^2-b^2}{2ca}, \qquad \cos C = \frac{a^2+b^2-c^2}{2ab}$$

Rearranged, the same rule gives a side from two sides and the included angle:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Special case. When $A = 90^\circ$ the cosine term vanishes and the rule collapses to Pythagoras, $a^2 = b^2 + c^2$.

03 PROJECTION FORMULA
A · 03

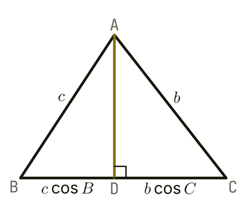
In $\triangle ABC$ let D be the foot of the perpendicular from A to BC. The projection of AB on BC is $BD = c\cos B$, and the projection of AC on BC is $CD = b\cos C$. Since $BC = BD + DC$,

$$a = c\cos B + b\cos C$$

and in the same way for the other two sides:

KEY

$$b = a\cos C + c\cos A, \qquad c = b\cos A + a\cos B$$



04 NAPIER'S ANALOGY
A · 04

Napier's formulae, also called the tangent rules, are:

KEY

$$\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}$$

$$\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\frac{B}{2}$$

$$\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$$

When to use them. These are the tool for the case where two sides and the included angle are given. The rule returns the difference of the other two angles; their sum is known, since $B + C = 180^\circ - A$, so both follow at once.

05 RATIOS OF THE HALF ANGLES
A · 05

With a, b, c the sides opposite A, B, C and the semi-perimeter

$$s = \frac{a+b+c}{2}$$

the half-angle ratios are

KEY

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}, \quad \sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}, \quad \cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}, \quad \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \quad \tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}, \quad \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

Signs. Each half angle lies in $(0^\circ, 90^\circ)$, so all three ratios are positive and the square roots are taken positive without further thought.

B AREA & THE CIRCLES 06–09

06 AREA OF A TRIANGLE
B · 06

Writing Δ for the area, the base and height form gives

$$\Delta = \tfrac{1}{2} \times \text{base} \times \text{height} = \tfrac{1}{2} BC \cdot AD$$

and replacing the height by a side times a sine gives the three symmetric forms

KEY

$$\Delta = \tfrac{1}{2}bc\sin A = \tfrac{1}{2}ca\sin B = \tfrac{1}{2}ab\sin C$$

HERON'S FORMULA

From the three sides alone,

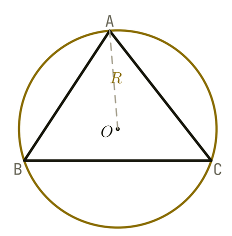
$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

07 CIRCUMCIRCLE AND CIRCUMRADIUS
B · 07

The circle passing through the three vertices is the circumcircle. Its centre, the circumcentre, is the point where the perpendicular bisectors of the sides meet, and its radius R is the circumradius:

KEY

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$$



WHERE THE CIRCUMCENTRE SITS

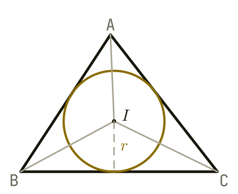
i. Its distances from the sides are $OD = R\cos A$, $OE = R\cos B$ and $OF = R\cos C$.
ii. For an acute-angled triangle it lies inside the triangle.
iii. For an obtuse-angled triangle it lies outside.
iv. For a right-angled triangle it is the mid-point of the hypotenuse.

08 INCIRCLE AND INRADIUS
B · 08

The circle inscribed in the triangle is the incircle. Its centre, the incentre, is the point where the internal bisectors of the angles meet, and its radius r is the inradius:

KEY

$$r = \frac{\Delta}{s}$$



Three further expressions for the same radius:

$$r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$$

$$r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = \frac{b\sin\frac{C}{2}\sin\frac{A}{2}}{\cos\frac{B}{2}} = \frac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}$$

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

09 ESCRIBED CIRCLES AND THE EXRADII
B · 09

The circle which touches BC and the other two sides produced is the escribed circle opposite A; its radius is written r_1, and its centre is an excentre. Similarly for r_2 and r_3.

KEY

$$r_1 = \frac{\Delta}{s-a}, \qquad r_2 = \frac{\Delta}{s-b}, \qquad r_3 = \frac{\Delta}{s-c}$$

$$r_1 = s\tan\frac{A}{2}, \qquad r_2 = s\tan\frac{B}{2}, \qquad r_3 = s\tan\frac{C}{2}$$

$$r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}, \quad r_2 = \frac{b\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}}, \quad r_3 = \frac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$$

$r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$, $\quad r_2 = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$, $\quad r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

TWO STANDARD RELATIONS

$$r_1 + r_2 + r_3 - r = 4R$$

$$r_1r_2 + r_2r_3 + r_3r_1 = \frac{r_1r_2r_3}{r} = s^2$$

C SOLVING A TRIANGLE 10–12

10 WHAT SOLVING A TRIANGLE MEANS
C · 10

A triangle has six elements: its three sides and its three angles. If three of them are given, at least one of which is a side, then the remaining three can be calculated. Finding them is called the solution of the triangle.

Why one side is needed. Three angles alone fix only the shape, not the size: infinitely many similar triangles share them. At least one length is required to pin the triangle down.

11 SOLVING A RIGHT-ANGLED TRIANGLE
C · 11

Let $\triangle ACB$ be right angled at C, so c is the hypotenuse.

WHEN TWO SIDES ARE GIVEN

GIVEN	TO BE CALCULATED
$a,\ b$	$\tan A = \frac{a}{b}$, $B = 90^\circ - A$, $c = \frac{a}{\sin A}$ or $c = \sqrt{a^2+b^2}$
$a,\ c$	$\sin A = \frac{a}{c}$, $B = 90^\circ - A$, $b = c\cos A$ or $b = \sqrt{c^2-a^2}$

WHEN ONE SIDE AND AN ACUTE ANGLE ARE GIVEN

GIVEN	TO BE CALCULATED
$a,\ A$	$B = 90^\circ - A$, $b = a\cot A$, $c = \frac{a}{\sin A}$
$c,\ A$	$B = 90^\circ - A$, $a = c\sin A$, $b = c\cos A$

12 SOLVING AN OBLIQUE TRIANGLE
C · 12

GIVEN	TO BE CALCULATED
Three sides a, b, c	$\sin A = \frac{2\Delta}{bc}$, $\sin B = \frac{2\Delta}{ca}$, $\sin C = \frac{2\Delta}{ab}$
Two sides and the included angle	$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$, then $a = \frac{b\sin A}{\sin B}$
One side and two angles	$C = 180^\circ - A - B$, then $b = \frac{a\sin B}{\sin A}$, $c = \frac{a\sin C}{\sin A}$
Two sides a, b and the opposite angle A	$\sin B = \frac{b\sin A}{a}$, then $c = \frac{a\sin C}{\sin A}$

The ambiguous case. The last row is the one to watch. If A is acute and $a < b\sin A$, then $\sin B > 1$, which is impossible, so no such triangle exists. Otherwise B may have two values, one acute and one obtuse, giving two different triangles.

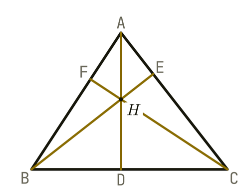
D CENTRES & RELATED TRIANGLES 13–17

13 ORTHOCENTRE
D · 13

The orthocentre H is the point where the three altitudes meet.

KEY

$$AH = 2R\cos A, \qquad BH = 2R\cos B, \qquad CH = 2R\cos C$$



Its distances from the feet of the altitudes are

$$DH = 2R\cos B\cos C, \quad EH = 2R\cos C\cos A, \quad FH = 2R\cos A\cos B$$

WHERE THE ORTHOCENTRE SITS

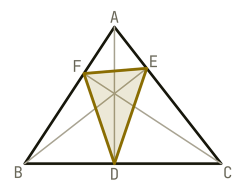
i. For an acute-angled triangle it lies inside; for an obtuse-angled triangle it lies outside.
ii. For a right-angled triangle it is the vertex containing the right angle.
iii. The image of H in any side of the triangle lies on the circumcircle, so $HD = DP$ where P is that image.

14 PEDAL TRIANGLE
D · 14

Joining the feet of the three altitudes gives the pedal triangle DEF.

KEY

$$EF = a\cos A, \qquad DF = b\cos B, \qquad DE = c\cos C$$



Its circumradius is exactly half that of the original triangle:

$$R' = \frac{R}{2}$$

A neat coincidence. In an acute-angled triangle the orthocentre of $\triangle ABC$ is the incentre of its pedal triangle $\triangle DEF$. The altitudes of the big triangle are the angle bisectors of the small one.

15 CENTROID AND THE MEDIANS
D · 15

The centroid G is the point where the three medians meet. It divides each median in the ratio $2 : 1$ from the vertex:

KEY

$$AG = \tfrac{2}{3}AD, \quad BG = \tfrac{2}{3}BE, \quad CG = \tfrac{2}{3}CF$$

$$DG = \tfrac{1}{3}AD, \quad EG = \tfrac{1}{3}BE, \quad FG = \tfrac{1}{3}CF$$

LENGTHS OF THE MEDIANS

$$AD = \tfrac{1}{2}\sqrt{2b^2+2c^2-a^2}, \qquad BE = \tfrac{1}{2}\sqrt{2a^2+2c^2-b^2}$$

$$CF = \tfrac{1}{2}\sqrt{2a^2+2b^2-c^2}$$

16 THE M-N THEOREM
D · 16

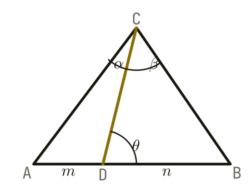
Let D be a point on AB dividing it into $AD = m$ and $DB = n$, and let CD split the angle at C into α and β, meeting AB at an angle θ. Then

KEY

$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$$

and, in terms of the base angles,

$$(m+n)\cot\theta = n\cot A - m\cot B$$



17 APOLLONIUS THEOREM
D · 17

If AD is a median of $\triangle ABC$, so that D is the mid-point of BC, then

KEY

$$AB^2 + AC^2 = 2\left(AD^2 + BD^2\right)$$

Link to the median formula. Putting $AB = c$, $AC = b$ and $BD = \frac{a}{2}$ and solving for AD gives exactly the median length $AD = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$ of section **15.**

E QUADRILATERALS & POLYGONS 18–19

18 RESULTS ON QUADRILATERALS
E · 18

For a general quadrilateral, the area follows from the diagonals and the angle α between them:

$$\text{area} = \tfrac{1}{2} \times (\text{product of the diagonals}) \times \sin\alpha$$

CYCLIC QUADRILATERALS

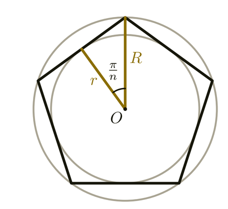
i. The sum of a pair of opposite angles is 180°.
ii. **Ptolemy's theorem.** In a cyclic quadrilateral the sum of the products of the two pairs of opposite sides equals the product of the diagonals.
iii. A circle can be inscribed in a quadrilateral if and only if the sums of its two pairs of opposite sides are equal.

19 REGULAR POLYGONS
E · 19

For a regular polygon of n sides, each of length a, let R be the radius of the circumscribed circle and r that of the inscribed circle. Then

KEY

$$R = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}, \qquad r = \frac{a}{2}\cot\frac{\pi}{n}$$



regular n-gon, here $n = 5$

The area of the polygon can be written three equivalent ways:

$$\text{area} = \tfrac{1}{4}na^2\cot\frac{\pi}{n} = nr^2\tan\frac{\pi}{n} = \tfrac{1}{2}nR^2\sin\frac{2\pi}{n}$$