

CONCEPT MAP

3D GEOMETRY

5 TOPICS • 34 CORE IDEAS • ONE SHEET



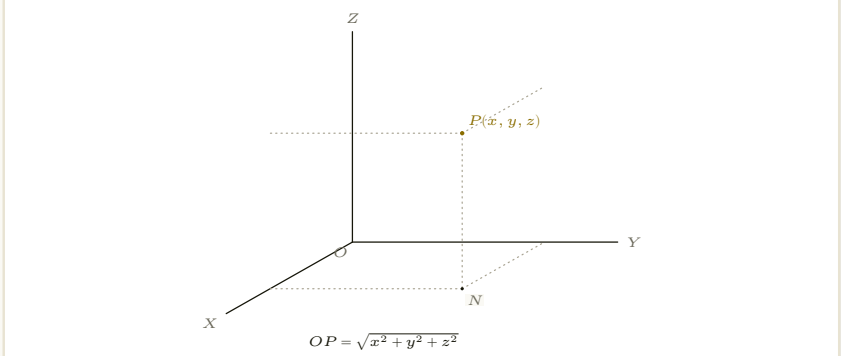
A POINTS & DIRECTIONS 01-06

01 COORDINATES IN SPACE A-01

Three mutually perpendicular axes through O fix every point by an ordered triple (x, y, z) .

KEY

$$OP = \sqrt{x^2 + y^2 + z^2}$$



- The three coordinate planes XY , YZ , ZX cut space into 8 octants.
- On the XY plane $z = 0$; on the X axis $y = z = 0$.
- Distance from the Z axis is $\sqrt{x^2 + y^2}$; distance from the XY plane is $|z|$.

Read the zeros. "Lies on the YZ plane" means $x = 0$, one equation. "Lies on the Y axis" means $x = 0$ and $z = 0$, two equations. Counting the conditions tells you how many unknowns are left.

02 DISTANCE AND SECTION FORMULA A-02

KEY

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The point dividing PQ in the ratio $m : n$ internally is

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n} \right)$$

- For external division replace n by $-n$.
- Mid-point: the average of the two triples.
- Centroid of a triangle: the average of the three vertices.
- Centroid of a tetrahedron: the average of the four vertices.

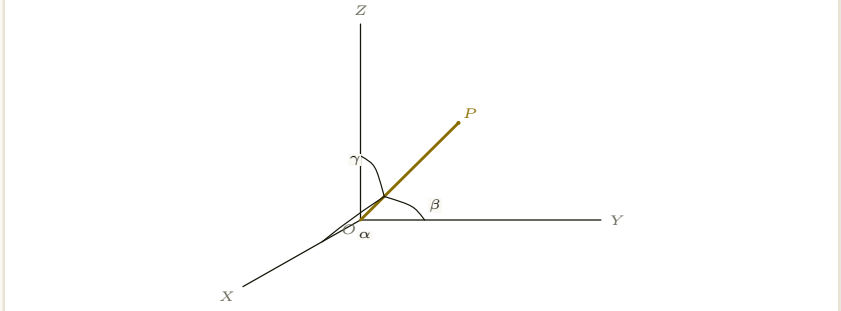
Finding a ratio. To find where PQ meets the plane $x = 0$, set the x coordinate of the section formula to zero and solve for $m : n$. A negative ratio simply means the point is outside the segment.

03 DIRECTION COSINES AND DIRECTION RATIOS A-03

If a directed line makes angles α, β, γ with the positive axes, its direction cosines are $l = \cos \alpha$, $m = \cos \beta$, $n = \cos \gamma$, and

KEY

$$l^2 + m^2 + n^2 = 1$$



Any triple proportional to (l, m, n) is a set of direction ratios (a, b, c) , and

$$l = \pm \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

Ratios are free, cosines are not. Direction ratios can be scaled by anything non-zero; direction cosines are fixed up to an overall sign, because the sum of their squares must be 1. Also $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$.

04 DIRECTION RATIOS FROM TWO POINTS A-04

The line through $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ has direction ratios

KEY

$$(x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

ANGLE BETWEEN TWO DIRECTIONS

$$\cos \theta = \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{\sum a_i^2} \sqrt{\sum a_j^2}}$$

- Perpendicular: $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$.
- Parallel: $b_1 \times b_2 = 0$, or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ when no denominator vanishes.

The modulus is not optional. A line has no direction, so the angle between two lines is taken as the acute one. Dropping the modulus gives the obtuse answer half the time.

05 PROJECTION OF A SEGMENT A-05

The projection of the segment joining P and Q on a line with direction cosines l, m, n is

KEY

$$l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

- In vector form this is $\vec{PQ} \cdot \hat{u}$: the component of $\vec{PQ}$ along the unit vector.
- Projections of $\vec{PQ}$ on the three axes are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$.
- The projection is zero exactly when PQ is perpendicular to the line.

Where it earns its keep. The perpendicular distance from a point to a line is $\sqrt{|\vec{AP}|^2 - (\text{projection of } \vec{AP})^2}$ — Pythagoras with the projection as the along-line leg, and no foot to compute.

06 THE VECTOR TOOLKIT A-06

KEY

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos \theta, \quad |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin \theta$$

OBJECT	WHAT IT GIVES
$\mathbf{a} \cdot \mathbf{b}$	angles, perpendicularity, projections
$\mathbf{a} \times \mathbf{b}$	a direction perpendicular to both
$[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$	signed volume; zero means coplanar
$ \mathbf{a} \times \mathbf{b} $	area of the parallelogram

Almost every result in this chapter is one of these three. A cross product whenever a perpendicular direction is wanted, a dot product whenever an angle or a distance is wanted, and a scalar triple product whenever the question is whether four points or two lines share a plane.

07 THE STRAIGHT LINE 07-14

07 EQUATION OF A LINE B-07

A line is fixed by one point on it and one direction along it. Through $A(a)$ with direction $\mathbf{b}$:

KEY

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}, \quad \lambda \in \mathbb{R}$$

In Cartesian form, with $A(x_1, y_1, z_1)$ and direction ratios (a, b, c) ,

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda$$

The two forms are the same object. Reading a Cartesian equation, the numerators give a point and the denominators give a direction. Anything not in that shape must be forced into it first — divide until each numerator is a bare $x - x_1$.

08 LINE THROUGH TWO POINTS B-08

KEY

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$$

and in Cartesian form,

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

- $\lambda = 0$ gives the first point, $\lambda = 1$ the second.
- $\lambda \in [0, 1]$ traces the segment; outside it, the rest of the line.
- Three points are collinear when the direction ratios of two of the joins are proportional.

09 WHEN A DENOMINATOR IS ZERO B-09

A zero in a denominator is not a division by zero; it is shorthand. If $c = 0$,

KEY

$$\frac{x - x_1}{a} = \frac{y - y_1}{b}, \quad z = z_1$$

The line is parallel to the XY plane. Two zeros mean the line is parallel to an axis.

Read it as a direction, not a fraction. The triple $(a, b, 0)$ is a perfectly good direction; it simply has no z component. Writing $\frac{z - z_1}{0}$ is the standard abuse of notation and means exactly $z = z_1$.

10 THE GENERAL POINT B-10

Every point of the line is

KEY

$$(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$$

with one unknown λ . This single move converts every geometric condition into an equation in λ .

One habit for the exam hall. Before computing anything, write the general point of every line the question mentions. It takes fifteen seconds and on nearly every question in this chapter the next step then becomes obvious. Students who skip it spend their time deciding what to do; students who do it spend their time doing it.

11 ANGLE BETWEEN TWO LINES B-11

KEY

$$\cos \theta = \frac{|\mathbf{b}_1 \cdot \mathbf{b}_2|}{|\mathbf{b}_1||\mathbf{b}_2|}$$

- Position never enters: only the two directions matter.
- Perpendicular: $\mathbf{b}_1 \cdot \mathbf{b}_2 = 0$.
- Parallel: $\mathbf{b}_1 \times \mathbf{b}_2 = 0$.
- Skew lines still have an angle between them, measured between their directions.

Two skew lines have an angle but no meeting point. The angle is defined by translating one direction to meet the other, which is why only $\mathbf{b}_1$ and $\mathbf{b}_2$ appear.

12 FOOT OF THE PERPENDICULAR B-12

Write F as the general point of the line, then force $\vec{PF}$ perpendicular to $\mathbf{b}$:

KEY

$$\vec{PF} \cdot \mathbf{b} = 0$$

KEY

$$\vec{PF} \cdot \mathbf{b} = 0$$

One linear equation gives λ , and the distance follows as $|\vec{PF}|$. Directly,

KEY

$$d = \frac{|\vec{AP} \times \mathbf{b}|}{|\mathbf{b}|}$$

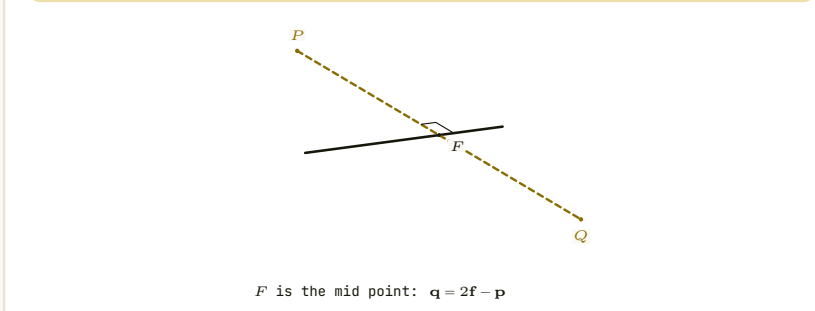
Two routes, one answer. Use the cross product when only the distance is wanted; go through the foot when the question also asks for the point, the image, or the equation of the perpendicular.

13 IMAGE OF A POINT IN A LINE B-13

The foot F is the mid-point of P and its image Q , so once F is known simply double:

KEY

$$\mathbf{f} = \frac{\mathbf{p} + \mathbf{q}}{2} \Rightarrow \mathbf{q} = 2\mathbf{f} - \mathbf{p}$$



Every mirror question is the same question. Foot, image, mirror, reflection: find F from $\vec{PF} \cdot \mathbf{b} = 0$, then double for an image. The word in the question changes; the work does not.

14 POINTS ON A LINE AT A GIVEN DISTANCE B-14

Take the general point, impose the distance, and square:

KEY

$$|\vec{AP}|^2 = d^2 \Rightarrow \text{a quadratic in } \lambda$$

- Two roots means two such points, one on each side.
- If $\mathbf{b}$ is a unit direction, λ is itself the signed distance from A , and the points are $\mathbf{a} \pm d\mathbf{b}$.
- Read the selection clause carefully: "on the same side as", "nearer the origin" or "with positive x " picks one root.

Normalise first and the quadratic disappears. With $|\mathbf{b}| = 1$ the two answers can be written down without solving anything.

C TWO LINES 15-20

15 THE THREE POSITIONS OF TWO LINES C-15

POSITION	TEST
Parallel	$\mathbf{b}_1 \times \mathbf{b}_2 = 0$
Intersecting	not parallel and $[\mathbf{a}_2 - \mathbf{a}_1, \mathbf{b}_1, \mathbf{b}_2] = 0$
Skew	not parallel and $[\mathbf{a}_2 - \mathbf{a}_1, \mathbf{b}_1, \mathbf{b}_2] \neq 0$

KEY

$$[\mathbf{a}_2 - \mathbf{a}_1, \mathbf{b}_1, \mathbf{b}_2] = (\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)$$

Test parallel first, always. If the two directions are proportional the triple product is zero automatically, and calling the lines intersecting on that basis is wrong — parallel lines are coplanar but never meet. The order of the two tests is the whole trick.

16 COPLANARITY AND INTERSECTION C-16

Two non-parallel lines are coplanar exactly when they meet, and the condition is

KEY

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

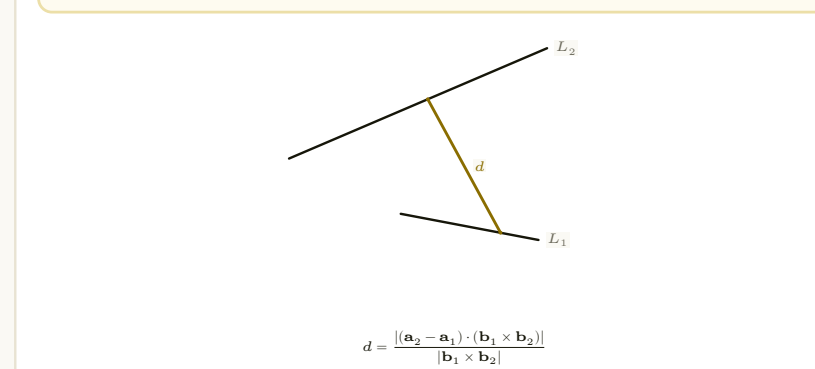
To find the point, equate the two general points: three equations in λ and μ .

Solve two, check the third. Two of the three coordinate equations fix λ and μ ; the third is the consistency test. If it fails the lines are skew, and that free check is what a "find k such that the lines intersect" question is really asking for.

17 SHORTEST DISTANCE BETWEEN SKEW LINES C-17

KEY

$$d = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|}$$



- $\mathbf{b}_1 \times \mathbf{b}_2$ is the direction of the common perpendicular.
- $d = 0$ is exactly the condition for the lines to intersect.
- The line of shortest distance meets both lines at right angles, and there is only one such line.

Why it is a projection. d is the component of the join $\mathbf{a}_2 - \mathbf{a}_1$ along the unit common normal. Every other join between the lines has the same component, which is why any $\mathbf{a}_1$ and $\mathbf{a}_2$ on the lines give the same answer.

18 DISTANCE BETWEEN PARALLEL LINES C-18

Here $\mathbf{b}_1 \times \mathbf{b}_2 = 0$, so the skew formula collapses. Use instead

KEY

$$d = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b}|}{|\mathbf{b}|}$$

which is just the distance from a point of one line to the other line.

The formula that fails is telling you something. A zero denominator in the skew formula is not an error; it is the detector for the parallel case. Switch formulas rather than pushing on.

19 DISTANCE MEASURED ALONG A LINE C-19

"The distance of P from the line L_1 , measured parallel to L_2 " is not a perpendicular distance. Write the line through P with direction $\mathbf{b}_2$, meet it with L_1 , and measure:

KEY

$$\mathbf{r} = \mathbf{p} + \mu \mathbf{b}_2 \text{ meets } \mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{b}_1$$

Two unknowns, three equations, one free check. Equating the two general points gives three scalar equations; solve two for λ and μ and let the third confirm the arithmetic. The answer is $|\mu||\mathbf{b}_2|$.

20 WHICH MOVE, WHEN C-20

THE QUESTION SAYS	THE MOVE
foot, image, mirror	$\vec{PF} \cdot \mathbf{b} = 0$; for an image, double
distance from a line	the foot, then $ \vec{PF} $; or the cross product
distance along a line	parametrise both, force them to meet
angle, perpendicular, parallel	directions only, with a modulus
shortest distance, skew	test parallel, then the triple product
coplanar, intersect, find k	set the triple product to zero
a point at a given distance	parametrise, square, solve a quadratic
line and plane	the only sine in the chapter

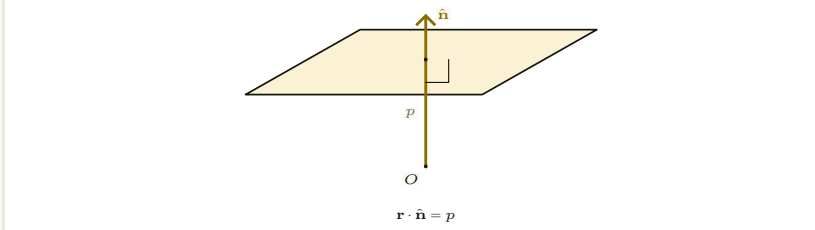
D THE PLANE 21-26

21 EQUATION OF A PLANE: NORMAL FORM D-21

A plane is fixed by a direction perpendicular to it and one number saying how far along that direction it sits. With $\hat{n}$ a unit normal and $p \geq 0$ the distance from O :

KEY

$$\mathbf{r} \cdot \hat{n} = p, \quad lx + my + nz = p$$



Normal form means unit normal. $2x + 2y + z = 6$ is not in normal form: divide by $|n| = 3$ to get $\frac{2}{3}x + \frac{2}{3}y + \frac{1}{3}z = 2$, and only then is the right-hand side the distance from the origin.

22 POINT-NORMAL AND GENERAL FORMS D-22

Through $A(a)$ with normal $\mathbf{n}$:

KEY

$$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$$
$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

Expanding gives the general first-degree equation

$$ax + by + cz + d = 0$$

Read the normal straight off. In $ax + by + cz + d = 0$ the coefficients (a, b, c) are the direction ratios of the normal. Every plane question starts by writing that triple down.

23 PLANE THROUGH THREE POINTS D-23

Three non-collinear points determine a plane. Either take $\mathbf{n} = \vec{AB} \times \vec{AC}$ and use the point-normal form, or write the condition directly:

KEY

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

INTERCEPT FORM

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

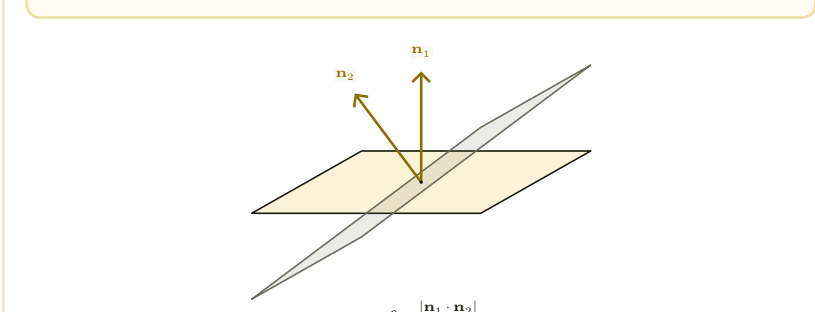
Four points are coplanar when the same determinant, built from the fourth point, vanishes — equivalently when the scalar triple product of the three edge vectors from one of them is zero.

24 ANGLE BETWEEN TWO PLANES D-24

The angle between two planes is the angle between their normals:

KEY

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}$$

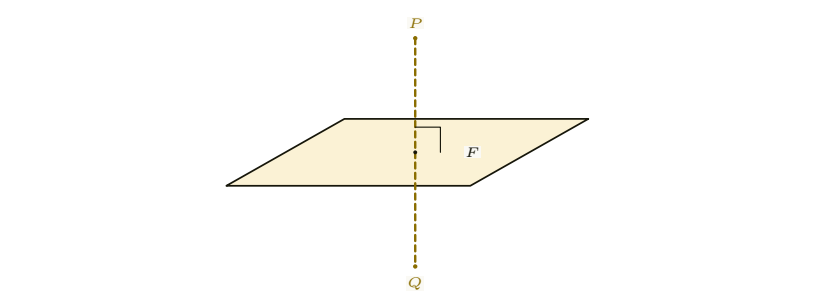


- Perpendicular planes: $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$.
- Parallel planes: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.
- Once rescaled to matching coefficients, two parallel planes differ only in the constant term.

25 DISTANCE OF A POINT FROM A PLANE D-25

KEY

$$d = \frac{|ax_1 + by_1 + cz_1 + d_0|}{\sqrt{a^2 + b^2 + c^2}}$$



Between the parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$,

$$d = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

Make the coefficients match first. The parallel-plane formula is only valid once both equations carry the same (a, b, c) . $2x + 2y + z = 3$ and $4x + 4y + 2z = 9$ must be rescaled before the constants can be subtracted.

26 FOOT AND IMAGE OF A POINT IN A PLANE D-26

The perpendicular from P to the plane has direction $\mathbf{n}$, so both the foot and the image lie on $\mathbf{r} = \mathbf{p} + t\mathbf{n}$. Writing $k = \frac{ax_1 + by_1 + cz_1 + d_0}{a^2 + b^2 + c^2}$,

KEY

$$\mathbf{f} = \mathbf{p} - k\mathbf{n}, \quad \mathbf{q} = \mathbf{p} - 2k\mathbf{n}$$

- The foot is halfway to the image, exactly as for a line.
- Two points are on the same side of the plane when $ax + by + cz + d_0$ has the same sign at both.
- The origin and P are on the same side when d_0 and that expression agree in sign.

27 FAMILY OF PLANES D-27

Every plane through the line of intersection of $P_1 = 0$ and $P_2 = 0$, with one exception, is

KEY

$$P_1 + \lambda P_2 = 0$$

- One condition on the plane gives one equation in λ .
- The family misses $P_2 = 0$ itself, which is the case $\lambda \rightarrow \infty$.
- Planes parallel to $ax + by + cz + d = 0$ are $ax + by + cz + k = 0$.

The standard three-mark question. "Find the plane through the line of intersection of two planes and through a given point" is: write $P_1 + \lambda P_2 = 0$, substitute the point, solve for λ , substitute back. Nothing else is needed.

28 BISECTORS OF TWO PLANES D-28

The two planes bisecting the angles between $P_1 = 0$ and $P_2 = 0$ are the locus of points equidistant from both:

KEY

$$\frac{a_1 x + b_1 y + c_1 z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2 x + b_2 y + c_2 z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Which sign is which. Make both constant terms positive first. The $+$ sign then gives the bisector of the angle containing the origin; the $-$ sign gives the other. The two bisector planes are always perpendicular to each other.

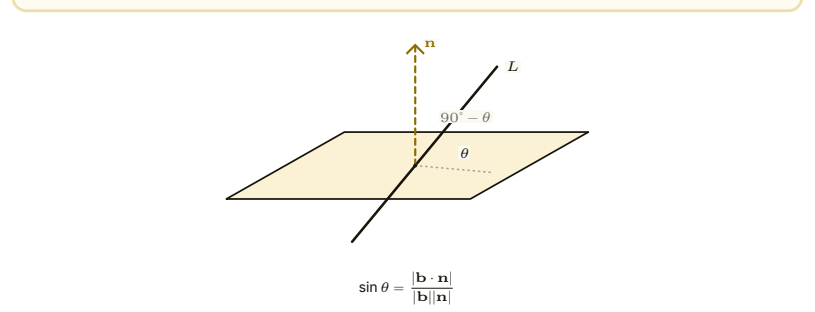
E LINE & PLANE TOGETHER 29-34

29 ANGLE BETWEEN A LINE AND A PLANE E-29

The line makes an angle $90^\circ - \theta$ with the normal, so the cosine becomes a sine:

KEY

$$\sin \theta = \frac{|\mathbf{b} \cdot \mathbf{n}|}{|\mathbf{b}||\mathbf{n}|}$$



The only sine in the chapter. Everywhere else — two lines, two planes — the formula is a cosine. If a question involves one line and one plane and you have written cos, you have the complement of the answer.