

[IIT-JEE MATHEMATICS]

JEE MAIN · ADVANCED

2027

MATHS

Formula Book

CLASS 11 AND 12

MATHEMATICS

CHAPTER WISE

IN THIS SYLLABUS

PAPER	MATHEMATICS
CHAPTERS	TWENTY THREE
SCOPE	FULL JEE MAIN AND ADVANCED
ALSO CARRIES	KEY RESULTS · MARK LOSERS
SEQUENCE	AS THE SYLLABUS IS TAUGHT
PAIRED WITH	THE COMPLETE SYLLABUS

[JEE MAIN AND ADVANCED]

How to use this book

This is a revision sheet, not a textbook. Nothing here is derived, because by the time you need a formula you should already know why it is true. Work the chapter first, then keep this beside you while you solve.

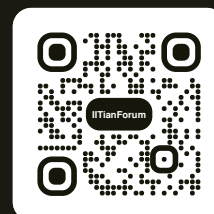
The chapters run in the sequence the syllabus is taught, so the book can be read straight through as a single revision pass, or opened at one chapter the night before a test. Each chapter opens with a dark bar carrying its number, and the contents overleaf gives the page. Chapter 11 is the longest, because inverse trigonometry is almost entirely formula and almost every mark lost in it is a range error rather than an algebra error.

Two kinds of card sit among the formulas. A gold card carries the results the paper leans on every year, the ones worth committing to memory. A dark card carries a place students reliably lose marks, usually a sign, a domain restriction, or a condition that is easy to drop. Neither is a substitute for the list beside it.

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SCAN FOR THE COURSE

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Twenty three chapters *in sequence*

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CHAPTER 01 BASIC ALGEBRA AND LOGARITHMS

IDENTITIES YOU SHOULD NEVER EXPAND BY HAND

- (i) $(a+b)^2 = a^2 + 2ab + b^2$, $(a-b)^2 = a^2 - 2ab + b^2$
- (ii) $a^2 - b^2 = (a+b)(a-b)$
- (iii) $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$
- (iv) $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
- (v) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+bc+ca)$

$$(vi) \quad a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

$$(vii) \quad a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2)$$

If $a+b+c=0$ then $a^3 + b^3 + c^3 = 3abc$. This one line closes a large family of questions.

DEFINITION · LOGARITHM

$$\log_a N = x \iff a^x = N$$

with $a > 0, a \neq 1, N > 0$

THE THREE LAWS

- (i) $\log_a(MN) = \log_a M + \log_a N$
- (ii) $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$
- (iii) $\log_a(M^p) = p \log_a M$
- (iv) $\log_{a^q}(M^p) = \frac{p}{q} \log_a M$

EVERY YEAR · CHANGE OF BASE

$$\log_a N = \frac{\log_b N}{\log_b a}$$

CONSEQUENCES OF THE CHANGE OF BASE

- (i) $\log_a b \cdot \log_b a = 1$
- (ii) $\log_a b \cdot \log_b c \cdot \log_c d = \log_a d$
- (iii) $a^{\log_a N} = N$
- (iv) $a^{\log_b c} = c^{\log_b a}$
- (v) $\log_a 1 = 0, \log_a a = 1$

WHERE MARKS GO · THE BASE DECIDES IT

$\log_a x > \log_a y$ means $x > y$ when $a > 1$, but $x < y$ when $0 < a < 1$. Solving a logarithmic inequality without splitting on the base is the standard way to lose the question. Check the domain first as well: every solution must keep every argument positive and every base positive and not 1.

CHARACTERISTIC AND MANTISSA

- (i) Digits in N to base ten: $\lfloor \log_{10} N \rfloor + 1$
- (ii) Zeros after the decimal in $N < 1$: $|\lfloor \log_{10} N \rfloor| - 1$
- (iii) $\log_{10} 2 \approx 0.3010, \log_{10} 3 \approx 0.4771, \log_{10} 7 \approx 0.8451$

MODULUS

- (i) $|x| = a \Rightarrow x = \pm a$ for $a \geq 0$; no solution for $a < 0$
- (ii) $|x| < a \iff -a < x < a$ ($a > 0$)
- (iii) $|x| > a \iff x < -a$ or $x > a$
- (iv) $||x| - |y|| \leq |x \pm y| \leq |x| + |y|$
- (v) $|xy| = |x||y|, \left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

POSITIVE REALS ONLY · MEANS

$$AM \geq GM \geq HM$$

equality exactly when all terms are equal

THE MEAN INEQUALITIES IN USE

- (i) $\frac{a_1 + \dots + a_n}{n} \geq (a_1 \dots a_n)^{1/n} \geq \frac{n}{\frac{1}{a_1} + \dots + \frac{1}{a_n}}$
- (ii) $GM^2 = AM \cdot HM$ for two numbers
- (iii) $x + \frac{1}{x} \geq 2$ for $x > 0$, and ≤ -2 for $x < 0$
- (iv) Cauchy: $(\sum a_i b_i)^2 \leq (\sum a_i^2)(\sum b_i^2)$

GREATEST INTEGER AND FRACTIONAL PART

- (i) $x = [x] + \{x\}, 0 \leq \{x\} < 1$
- (ii) $[x] \leq x < [x] + 1, x - 1 < [x] \leq x$
- (iii) $[x + n] = [x] + n$ for integer n
- (iv) $[x] + [-x] = 0$ if $x \in \mathbb{Z}$, else -1

CHAPTER 02 SETS, RELATIONS AND FUNCTIONS

COUNTING IN SETS

- (i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- (ii) $n(A \cup B \cup C) = \sum n(A) - \sum n(A \cap B) + n(A \cap B \cap C)$
- (iii) A set with n elements has 2^n subsets and $2^n - 1$ proper subsets

- (iv) Relations from A to B : $2^{n(A)n(B)}$

De Morgan: $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$.

TYPES OF RELATION ON N ELEMENTS

TYPE	COUNT
Reflexive	2^{n^2-n}
Symmetric	$2^{n(n+1)/2}$
Reflexive and symmetric	$2^{n(n-1)/2}$
Equivalence	Bell number B_n

Transitive relations have no closed formula. An equivalence relation is exactly a partition of the set, which is how the counting questions are meant to be read.

COUNTING FUNCTIONS FROM A TO B

With $|A| = m$ and $|B| = n$:

- (i) Total: n^m
- (ii) One-one: ${}^nP_m$ if $m \leq n$, else 0
- (iii) Onto: $\sum_{r=0}^n (-1)^r {}^nC_r (n-r)^m$
- (iv) Bijections: $n!$ if $m = n$, else 0

DOMAIN AND RANGE

$F(X)$	DOMAIN	RANGE
$\sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$1/x$	$\mathbb{R} - \{0\}$	$\mathbb{R} - \{0\}$
$\log x$	$(0, \infty)$	$\mathbb{R}$
e^x	$\mathbb{R}$	$(0, \infty)$
$[x]$	$\mathbb{R}$	$\mathbb{Z}$
$\{x\}$	$\mathbb{R}$	$[0, 1)$
$\operatorname{sgn} x$	$\mathbb{R}$	$\{-1, 0, 1\}$

EVEN, ODD AND PERIODIC

- (i) Even: $f(-x) = f(x)$. Odd: $f(-x) = -f(x)$
- (ii) Every f splits as $\frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$
- (iii) If $f(x+T) = f(x)$, the period of $f(ax+b)$ is $T/|a|$
- (iv) Period of $\sin x, \cos x$ is 2π ; of $\tan x, \cot x$ is π
- (v) Period of $|\sin x|$ is π ; of $\{x\}$ is 1
- (vi) The period of a sum is the LCM of the periods, when it exists

WHERE MARKS GO · INVERSE NEEDS A BIJECTION

Check one-one and onto on the stated codomain before writing f^{-1} . Note also that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$: the order reverses.

FUNCTIONAL EQUATIONS AND THEIR ANSWERS

GIVEN	FORM OF F
$f(x+y) = f(x) + f(y)$	kx
$f(xy) = f(x) + f(y)$	$k \log x$
$f(x+y) = f(x)f(y)$	a^x
$f(xy) = f(x)f(y)$	x^n
$f(x)f(1/x) = f(x) + f(1/x)$	$1 \pm x^n$

CHAPTER 03 QUADRATIC EQUATIONS

FOR $AX^2 + BX + C = 0$ · ROOTS

$$x = \frac{-b \pm \sqrt{D}}{2a}, \quad D = b^2 - 4ac$$

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}$$

THE DISCRIMINANT

D	ROOTS, REAL COEFFICIENTS
> 0	real and distinct
$= 0$	real and equal, $-b/2a$
< 0	conjugate complex pair
perfect square	rational, if $a, b, c \in \mathbb{Q}$

$$|\alpha - \beta| = \frac{\sqrt{|D|}}{|a|}.$$

WHERE MARKS GO · CONJUGATE ROOTS NEED REALS

Surd roots pair as $p \pm \sqrt{q}$ only when the coefficients are rational, and complex roots pair as $z, \bar{z}$ only when they are real. Drop that condition and the whole argument fails.

BUILDING THE EQUATION YOU WANT

- (i) Roots α, β : $x^2 - (\alpha + \beta)x + \alpha\beta = 0$
- (ii) Roots $\frac{1}{\alpha}, \frac{1}{\beta}$: reverse the coefficients, $cx^2 + bx + a = 0$
- (iii) Roots $-\alpha, -\beta$: $ax^2 - bx + c = 0$
- (iv) Roots $\alpha + k, \beta + k$: replace x by $x - k$
- (v) Roots $k\alpha, k\beta$: $ax^2 + kbx + k^2c = 0$

SYMMETRIC FUNCTIONS

- (i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
- (ii) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
- (iii) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$

SIGN OF THE QUADRATIC

- (i) $ax^2 + bx + c > 0$ for all $x \iff a > 0$ and $D < 0$
- (ii) $ax^2 + bx + c < 0$ for all $x \iff a < 0$ and $D < 0$
- (iii) $ax^2 + bx + c \geq 0$ for all $x \iff a > 0$ and $D \leq 0$
- (iv) Between the roots the sign is opposite to a ; outside, it agrees with a

- (v) Extreme value $\frac{4ac - b^2}{4a}$ at $x = -\frac{b}{2a}$, a minimum if $a > 0$

LOCATION OF ROOTS, $F(X) = AX^2 + BX + C$

REQUIREMENT	CONDITIONS
Both roots $> k$	$D \geq 0, af(k) > 0, -\frac{b}{2a} > k$
Both roots $< k$	$D \geq 0, af(k) > 0, -\frac{b}{2a} < k$
k lies between	$af(k) < 0$
Both in (k_1, k_2)	$D \geq 0, af(k_1) > 0, af(k_2) > 0, k_1 < -\frac{b}{2a} < k_2$
Exactly one in (k_1, k_2)	$f(k_1)f(k_2) < 0$

COMMON ROOTS

- (i) One common root of $a_1x^2 + b_1x + c_1$ and $a_2x^2 + b_2x + c_2$:
 $(a_1c_2 - a_2c_1)^2 = (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1)$
- (ii) That root is $\frac{a_1c_2 - a_2c_1}{a_2b_1 - a_1b_2}$
- (iii) Both roots common: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

CUBIC AND HIGHER

- (i) For $ax^3 + bx^2 + cx + d = 0$: $\sum \alpha = -\frac{b}{a}, \sum \alpha\beta = \frac{c}{a}, \alpha\beta\gamma = -\frac{d}{a}$
- (ii) Descartes: sign changes in $f(x)$ bound the positive roots, in $f(-x)$ the negative
- (iii) A polynomial of odd degree with real coefficients has at least one real root

CHAPTER 04 COMPLEX NUMBERS

MODULUS, CONJUGATE, ARGUMENT

- (i) $z = x + iy, |z| = \sqrt{x^2 + y^2}, \bar{z} = x - iy$
- (ii) $z\bar{z} = |z|^2, z^{-1} = \frac{\bar{z}}{|z|^2}$
- (iii) $|z_1z_2| = |z_1||z_2|, \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- (iv) $\arg(z_1z_2) = \arg z_1 + \arg z_2$ modulo 2π
- (v) $\operatorname{Re} z = \frac{z + \bar{z}}{2}, \operatorname{Im} z = \frac{z - \bar{z}}{2i}$
- (vi) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$
- (vii) $||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$

EVERY YEAR · POLAR FORM AND DE MOIVRE

$$z = r(\cos \theta + i \sin \theta) = re^{i\theta}$$

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

ROOTS

- (i) n th roots of $z = re^{i\theta}$: $r^{1/n}e^{i(\theta+2k\pi)/n}, k = 0, \dots, n-1$
- (ii) The n th roots of unity sum to 0 and have product $(-1)^{n+1}$

(iii) They are the vertices of a regular n -gon on the unit circle

(iv) $\sqrt{a+ib} = \pm \left(\sqrt{\frac{|z|+a}{2}} + i \operatorname{sgn}(b) \sqrt{\frac{|z|-a}{2}} \right)$

(v) $i^{4k} = 1, i^{4k+1} = i, i^{4k+2} = -1, i^{4k+3} = -i$

MEMORISE · CUBE ROOTS OF UNITY

$$\omega = \frac{-1+i\sqrt{3}}{2}, \quad \omega^3 = 1$$

$$1 + \omega + \omega^2 = 0$$

WORKING WITH ω

(i) $\omega^2 = \bar{\omega} = \frac{1}{\omega}$

(ii) $a^3 + b^3 = (a+b)(a+b\omega)(a+b\omega^2)$

(iii) $a^2 + b^2 + c^2 - ab - bc - ca = (a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$

(iv) $1 + \omega^n + \omega^{2n} = 3$ if $3 \mid n$, else 0

LOCI IN THE PLANE

EQUATION	LOCUS
$ z - z_0 = r$	circle, centre z_0 , radius r
$ z - z_1 = z - z_2 $	perpendicular bisector
$ z - z_1 + z - z_2 = k$	ellipse if $k > z_1 - z_2 $
$ z - z_1 - z - z_2 = k$	hyperbola if $k < z_1 - z_2 $
$\left \frac{z - z_1}{z - z_2} \right = k \neq 1$	circle of Apollonius
$\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha$	arc through z_1, z_2

WHERE MARKS GO · THE DEGENERATE CASES

$|z - z_1| + |z - z_2| = |z_1 - z_2|$ is the segment, not an ellipse. An argument of $\pi/2$ gives the circle on $z_1 z_2$ as diameter, and an argument of 0 or π gives a line. The paper prefers exactly these edges.

THE ONE GEOMETRY FORMULA · ROTATION

$$\frac{z_3 - z_2}{z_1 - z_2} = \left| \frac{z_3 - z_2}{z_1 - z_2} \right| e^{i\theta}$$

with θ measured at z_2

TRIANGLES IN THE PLANE

(i) Collinear $\iff \frac{z_3 - z_1}{z_2 - z_1} \in \mathbb{R}$

(ii) Equilateral $\iff z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$

(iii) Area $= \frac{1}{2} |\operatorname{Im}(\bar{z}_1 z_2 + \bar{z}_2 z_3 + \bar{z}_3 z_1)|$

CHAPTER 05 SEQUENCES AND SERIES

MEMORISE · ARITHMETIC PROGRESSION

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2} [2a + (n-1)d] = \frac{n}{2} (a + \ell)$$

WORKING WITH AN AP

(i) $T_n = S_n - S_{n-1}$

(ii) Any term is the AM of its neighbours: $2T_n = T_{n-1} + T_{n+1}$

(iii) The sum of terms equidistant from the ends is constant

(iv) n AMs between a and b have common difference $\frac{b-a}{n+1}$ and sum $\frac{n(a+b)}{2}$

MEMORISE · GEOMETRIC PROGRESSION

$$T_n = ar^{n-1}, \quad S_n = a \frac{r^n - 1}{r - 1} \quad (r \neq 1)$$

$$S_\infty = \frac{a}{1-r} \quad \text{for } |r| < 1$$

WORKING WITH A GP

(i) $T_n^2 = T_{n-1} T_{n+1}$

- (ii) n GMs between a and b have ratio $(b/a)^{1/(n+1)}$ and product $(ab)^{n/2}$

WHERE MARKS GO · CHECK THE RATIO FIRST

$S_\infty = a/(1-r)$ is meaningless unless $|r| < 1$. A question that hands you r in terms of a parameter is usually testing exactly that range, not the sum.

HARMONIC PROGRESSION

- (i) $a_1, a_2, \dots$ is an HP when $\frac{1}{a_1}, \frac{1}{a_2}, \dots$ is an AP
- (ii) $T_n = \frac{1}{a + (n-1)d}$
- (iii) HM of a and b is $\frac{2ab}{a+b}$
- (iv) $A \geq G \geq H$ and $G^2 = AH$ for two positive numbers

There is no formula for the sum of an HP. Questions asking for one are asking you to convert to an AP first.

EVERY YEAR · STANDARD SUMS

$$\Sigma n = \frac{n(n+1)}{2}, \quad \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\Sigma n^3 = \left[\frac{n(n+1)}{2} \right]^2 = (\Sigma n)^2$$

TWO MORE SUMS

- (i) $\Sigma(2n-1) = n^2, \quad \Sigma 2n = n(n+1)$
- (ii) $\Sigma n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

ARITHMETICO-GEOMETRIC SERIES

- (i) Terms $[a + (n-1)d]r^{n-1}$; for $|r| < 1$, $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$
- (ii) Finite case: multiply by r , subtract, then sum the resulting GP
- (iii) $\sum_{n=1}^{\infty} \frac{n}{2^n} = 2, \quad \sum_{n=1}^{\infty} \frac{n^2}{2^n} = 6$

TELESCOPING

- (i) $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, and the sum to n is $\frac{n}{n+1}$
- (ii) $\frac{1}{n(n+1)(n+2)} = \frac{1}{2} \left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \right]$
- (iii) $\frac{1}{\sqrt{n} + \sqrt{n+1}} = \sqrt{n+1} - \sqrt{n}$
- (iv) $\tan^{-1} \frac{1}{1+n+n^2} = \tan^{-1}(n+1) - \tan^{-1} n$

EXPONENTIAL AND LOGARITHMIC SERIES

- (i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$ for all x
- (ii) $e = \sum_{n=0}^{\infty} \frac{1}{n!}, \quad e + e^{-1} = 2 \sum_{n=0}^{\infty} \frac{1}{(2n)!}$
- (iii) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots, \quad -1 < x \leq 1$
- (iv) $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots, \quad -1 \leq x < 1$

CHAPTER 06 PERMUTATIONS AND COMBINATIONS

ORDER MATTERS OR NOT · THE TWO COUNTS

$${}^n P_r = \frac{n!}{(n-r)!}, \quad {}^n C_r = \frac{n!}{r!(n-r)!}$$

IDENTITIES

- (i) ${}^n C_r = {}^n C_{n-r}, \quad {}^n C_r + {}^n C_{r-1} = {}^{n+1} C_r$
- (ii) $r \cdot {}^n C_r = n \cdot {}^{n-1} C_{r-1}$
- (iii) ${}^n C_x = {}^n C_y \Rightarrow x = y \text{ or } x + y = n$
- (iv) ${}^n P_r = r! \cdot {}^n C_r$

ARRANGEMENTS WITH REPETITION

- (i) r places, n choices each, repetition allowed: n^r
- (ii) n objects with p, q, r alike: $\frac{n!}{p!q!r!}$
- (iii) Selections from n alike objects: $n+1$ ways, including none
- (iv) From p alike, q alike and n distinct: $(p+1)(q+1)2^n - 1$

CIRCULAR ARRANGEMENTS

- (i) n distinct around a circle: $(n-1)!$

- (ii) Clockwise and anticlockwise the same: $\frac{(n-1)!}{2}$
 (iii) r chosen from n and seated in a circle: $\frac{{}^nP_r}{r}$

DISTRIBUTION AND DIVISION

- (i) n identical into r distinct boxes, none empty:
 ${}^{n-1}C_{r-1}$
 (ii) Same, empty allowed: ${}^{n+r-1}C_{r-1}$
 (iii) n distinct into r distinct boxes, empty allowed: r^n
 (iv) n distinct into r distinct boxes, none empty:
 $\sum_{k=0}^r (-1)^k {}^rC_k (r-k)^n$
 (v) $2n$ distinct into two equal unnamed groups:
 $\frac{(2n)!}{2!(n!)^2}$

WHERE MARKS GO · NAMED OR UNNAMED GROUPS

Dividing into groups of equal size needs a division by $k!$ when the k groups are unnamed, and no such division when they are named. This single factor is the most frequent error in the chapter.

DERANGEMENTS AND STANDARD COUNTS

- (i) $D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!}$; $D_1 = 0$, $D_2 = 1$, $D_3 = 2$,

$$D_4 = 9, D_5 = 44$$

- (ii) Exactly r fixed points: ${}^nC_r D_{n-r}$
 (iii) Divisors of $N = p^a q^b r^c$: $(a+1)(b+1)(c+1)$
 (iv) Sum of divisors: $\frac{p^{a+1}-1}{p-1} \cdot \frac{q^{b+1}-1}{q-1} \dots$
 (v) Exponent of a prime p in $n!$: $\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \dots$

GEOMETRY COUNTS

From n points, no three collinear:

- (i) Lines ${}^nC_2$, triangles ${}^nC_3$
 (ii) Diagonals of an n -gon: $\frac{n(n-3)}{2}$
 (iii) Rectangles in an $m \times n$ grid: ${}^{m+1}C_2 \cdot {}^{n+1}C_2$
 (iv) Squares in an $n \times n$ grid: $\frac{n(n+1)(2n+1)}{6}$

If p of the n points are collinear, subtract ${}^pC_2 - 1$ from the lines and ${}^pC_3$ from the triangles.

GAP METHOD

- (i) m items, then n more with no two of the n together:
 $m! \cdot {}^{m+1}P_n$
 (ii) Non-consecutive selection of r from n in a row:
 ${}^{n-r+1}C_r$

CHAPTER 07 BINOMIAL THEOREM

POSITIVE INTEGRAL INDEX · THE EXPANSION

$$(x+a)^n = \sum_{r=0}^n {}^nC_r x^{n-r} a^r$$

$$T_{r+1} = {}^nC_r x^{n-r} a^r$$

READING THE EXPANSION

- (i) There are $n+1$ terms; powers of x fall, powers of a rise
 (ii) T_{r+1} from the end = T_{n-r+1} from the start
 (iii) The general term of $(x-a)^n$ carries $(-1)^r$

WHERE MARKS GO · R AGAINST R+1

The coefficient ${}^nC_r$ sits in the $(r+1)$ th term. Solving for the term number and then substituting it as r is the classic off-by-one in this chapter.

MIDDLE AND GREATEST TERMS

- (i) n even: one middle term, $T_{\frac{n}{2}+1}$
 (ii) n odd: two middle terms, $T_{\frac{n+1}{2}}$ and $T_{\frac{n+3}{2}}$
 (iii) Greatest coefficient: ${}^nC_{n/2}$ for even n ; ${}^nC_{(n\pm 1)/2}$ for odd n
 (iv) Numerically greatest term: with $k = \frac{(n+1)|a|}{|x|+|a|}$, take T_k and T_{k+1} if k is an integer, else $T_{[k]+1}$

SUMS OF COEFFICIENTS

- (i) $\sum {}^nC_r = 2^n$, $\sum (-1)^r {}^nC_r = 0$
 (ii) ${}^nC_0 + {}^nC_2 + \dots = {}^nC_1 + {}^nC_3 + \dots = 2^{n-1}$
 (iii) $\sum r {}^nC_r = n2^{n-1}$
 (iv) $\sum r^2 {}^nC_r = n(n+1)2^{n-2}$
 (v) $\sum_r \frac{{}^nC_r}{r+1} = \frac{2^{n+1}-1}{n+1}$

$$(vi) \sum \binom{n}{r} C_r^2 = {}^{2n}C_n$$

Put $x = 1$ for the total, $x = -1$ for the alternating sum, differentiate for a factor of r , integrate for a factor of $1/(r+1)$.

VANDERMONDE

$$(i) \sum_r {}^m C_r {}^n C_{p-r} = {}^{m+n}C_p$$

$$(ii) \sum {}^n C_r {}^n C_{r+k} = {}^{2n}C_{n-k}$$

VALID ONLY FOR $|X| < 1$ · ANY INDEX

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$$

for any real n

THE FOUR YOU SHOULD RECOGNISE

$$(i) (1-x)^{-1} = 1 + x + x^2 + \dots$$

$$(ii) (1+x)^{-1} = 1 - x + x^2 - \dots$$

$$(iii) (1-x)^{-2} = 1 + 2x + 3x^2 + \dots$$

$$(iv) (1-x)^{-n} = \sum_{r \geq 0} {}^{n+r-1}C_r x^r$$

MULTINOMIAL

$$(i) (x_1 + \dots + x_k)^n = \sum \frac{n!}{r_1! \dots r_k!} x_1^{r_1} \dots x_k^{r_k}, \text{ with } \sum r_i = n$$

$$(ii) \text{Number of terms: } {}^{n+k-1}C_{k-1}$$

DIVISIBILITY AND REMAINDER

(i) Write the base as a multiple of the divisor plus or minus a small number, then expand

(ii) $(1+x)^n - 1$ is divisible by x ; $(1+x)^n - 1 - nx$ by x^2

(iii) Last-digit questions reduce to a cycle of length 1, 2 or 4

CHAPTER 08 MATRICES AND DETERMINANTS

DETERMINANT ESSENTIALS

$$(i) |AB| = |A||B|, |A^T| = |A|, |kA| = k^n |A| \text{ for order } n$$

$$(ii) |A^{-1}| = \frac{1}{|A|}, |\text{adj } A| = |A|^{n-1}$$

$$(iii) |\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$$

(iv) Swapping two rows changes the sign; two equal rows give 0

(v) Adding a multiple of one row to another leaves $|A|$ unchanged

EVERY YEAR · INVERSE AND ADJOINT

$$A(\text{adj } A) = (\text{adj } A)A = |A|I$$

$$A^{-1} = \frac{\text{adj } A}{|A|}, \text{ exists iff } |A| \neq 0$$

PRODUCTS AND TRANSPOSES

$$(i) \text{adj}(AB) = (\text{adj } B)(\text{adj } A)$$

$$(ii) \text{adj}(\text{adj } A) = |A|^{n-2}A$$

$$(iii) (AB)^{-1} = B^{-1}A^{-1}, (A^T)^{-1} = (A^{-1})^T$$

$$(iv) \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

SPECIAL MATRICES

NAME	CONDITION
Symmetric	$A^T = A$
Skew symmetric	$A^T = -A$, diagonal all 0
Orthogonal	$AA^T = I$, so $ A = \pm 1$
Idempotent	$A^2 = A$
Involutory	$A^2 = I$
Nilpotent	$A^k = O$ for some k
Singular	$ A = 0$

Every square matrix is $\frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$, symmetric plus skew symmetric. A skew symmetric matrix of odd order has determinant 0.

SYSTEMS OF LINEAR EQUATIONS

CASE	CONCLUSION
$\Delta \neq 0$	unique, $x_i = \Delta_i / \Delta$
$\Delta = 0$, some $\Delta_i \neq 0$	inconsistent
$\Delta = 0$, all $\Delta_i = 0$	infinitely many, or inconsistent

A homogeneous system always has the trivial solution, and has a non-trivial one exactly when $\Delta = 0$.

WHERE MARKS GO · ALL $\Delta_i = 0$ IS NOT ENOUGH

With $\Delta = 0$ and every $\Delta_i = 0$ the system may still be inconsistent, for example two parallel planes and a third cutting both. Verify by elimination rather than declaring infinitely many solutions.

USEFUL DETERMINANTS

$$(i) \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

$$(ii) \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$$

$$(iii) \text{Area of a triangle} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}, \text{ zero exactly when the points are collinear}$$

TRACE, POWERS, CAYLEY HAMILTON

$$(i) \operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B, \operatorname{tr}(AB) = \operatorname{tr}(BA)$$

$$(ii) \text{For } 2 \times 2: A^2 - (\operatorname{tr} A)A + |A|I = O$$

(iii) That identity gives A^{-1} and every high power without multiplying matrices

$$(iv) \text{If } A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ then } A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

CHAPTER 09 TRIGONOMETRIC FUNCTIONS AND IDENTITIES

MEMORISE · THE THREE IDENTITIES

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ 1 + \tan^2 \theta &= \sec^2 \theta \\ 1 + \cot^2 \theta &= \operatorname{cosec}^2 \theta \end{aligned}$$

TWO CONSEQUENCES

$$\begin{aligned} (i) \sec \theta + \tan \theta &= \frac{1}{\sec \theta - \tan \theta} \\ (ii) \operatorname{cosec} \theta + \cot \theta &= \frac{1}{\operatorname{cosec} \theta - \cot \theta} \end{aligned}$$

DOMAIN AND RANGE

	DOMAIN	RANGE
$\sin \theta$	$\mathbb{R}$	$[-1, 1]$
$\cos \theta$	$\mathbb{R}$	$[-1, 1]$
$\tan \theta$	$\mathbb{R} - \{(2n+1)\frac{\pi}{2}\}$	$\mathbb{R}$
$\cot \theta$	$\mathbb{R} - \{n\pi\}$	$\mathbb{R}$
$\sec \theta$	$\mathbb{R} - \{(2n+1)\frac{\pi}{2}\}$	$\mathbb{R} - (-1, 1)$
$\operatorname{cosec} \theta$	$\mathbb{R} - \{n\pi\}$	$\mathbb{R} - (-1, 1)$

COMPOUND ANGLES

$$(i) \sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$(ii) \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$(iii) \tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$(iv) \sin(A+B) \sin(A-B) = \sin^2 A - \sin^2 B$$

$$(v) \cos(A+B) \cos(A-B) = \cos^2 A - \sin^2 B$$

MULTIPLE ANGLES

$$\begin{aligned}
 \text{(i)} \quad \sin 2\theta &= 2 \sin \theta \cos \theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} \\
 \text{(ii)} \quad \cos 2\theta &= 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \\
 \text{(iii)} \quad \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta} \\
 \text{(iv)} \quad \sin 3\theta &= 3 \sin \theta - 4 \sin^3 \theta \\
 \text{(v)} \quad \cos 3\theta &= 4 \cos^3 \theta - 3 \cos \theta \\
 \text{(vi)} \quad \tan 3\theta &= \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}
 \end{aligned}$$

HALF ANGLES

$$\begin{aligned}
 \text{(i)} \quad \sin^2 \frac{\theta}{2} &= \frac{1 - \cos \theta}{2}, \quad \cos^2 \frac{\theta}{2} = \frac{1 + \cos \theta}{2} \\
 \text{(ii)} \quad \tan \frac{\theta}{2} &= \frac{\sin \theta}{1 + \cos \theta} = \frac{1 - \cos \theta}{\sin \theta} \\
 \text{(iii)} \quad \text{With } t = \tan \frac{\theta}{2}: \sin \theta &= \frac{2t}{1 + t^2}, \quad \cos \theta = \frac{1 - t^2}{1 + t^2}
 \end{aligned}$$

SUM TO PRODUCT, PRODUCT TO SUM

$$\begin{aligned}
 \text{(i)} \quad \sin C + \sin D &= 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2} \\
 \text{(ii)} \quad \sin C - \sin D &= 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2} \\
 \text{(iii)} \quad \cos C + \cos D &= 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2} \\
 \text{(iv)} \quad \cos C - \cos D &= -2 \sin \frac{C+D}{2} \sin \frac{C-D}{2} \\
 \text{(v)} \quad 2 \sin A \cos B &= \sin(A+B) + \sin(A-B) \\
 \text{(vi)} \quad 2 \cos A \cos B &= \cos(A+B) + \cos(A-B) \\
 \text{(vii)} \quad 2 \sin A \sin B &= \cos(A-B) - \cos(A+B)
 \end{aligned}$$

SPECIAL ANGLES

$$\text{(i)} \quad \sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^\circ$$

$$\text{(ii)} \quad \cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75^\circ$$

$$\text{(iii)} \quad \tan 15^\circ = 2 - \sqrt{3}, \quad \tan 75^\circ = 2 + \sqrt{3}, \quad \tan 22\frac{1}{2}^\circ = \sqrt{2} - 1$$

$$\text{(iv)} \quad \sin 18^\circ = \frac{\sqrt{5}-1}{4} = \cos 72^\circ$$

$$\text{(v)} \quad \cos 36^\circ = \frac{\sqrt{5}+1}{4} = \sin 54^\circ$$

$$\text{(vi)} \quad \cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}, \quad \sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

WHERE MARKS GO · TWO SHAPES, NOT ONE

$\cos 18^\circ$ carries a nested radical and $\cos 36^\circ$ does not. Writing $\cos 18^\circ = \frac{\sqrt{5}+1}{4}$ is the commonest slip in this list.

EVERY YEAR · GENERAL SOLUTIONS

$$\sin \theta = \sin \alpha \Rightarrow \theta = n\pi + (-1)^n \alpha$$

$$\cos \theta = \cos \alpha \Rightarrow \theta = 2n\pi \pm \alpha$$

$$\tan \theta = \tan \alpha \Rightarrow \theta = n\pi + \alpha$$

SQUARED AND SPECIAL CASES

$$\text{(i)} \quad \sin^2 \theta = \sin^2 \alpha, \quad \cos^2 \theta = \cos^2 \alpha \text{ or } \tan^2 \theta = \tan^2 \alpha \Rightarrow \theta = n\pi \pm \alpha$$

$$\text{(ii)} \quad \sin \theta = 0 \Rightarrow \theta = n\pi; \quad \cos \theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2}$$

EVERY YEAR · MAXIMUM AND MINIMUM

$$-\sqrt{a^2 + b^2} \leq a \sin x + b \cos x \leq \sqrt{a^2 + b^2}$$

The bounds are attained, so they are the actual maximum and minimum. For $a \sin^2 x + b \cos^2 x$ the range is the interval between a and b .

CHAPTER 10 PROPERTIES OF TRIANGLES

MEMORISE · THE TWO RULES

$$\begin{aligned}
 \frac{a}{\sin A} &= \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \\
 \cos A &= \frac{b^2 + c^2 - a^2}{2bc}
 \end{aligned}$$

TWO MORE RELATIONS

$$\text{(i)} \quad \text{Projection: } a = b \cos C + c \cos B$$

$$\text{(ii)} \quad \text{Napier: } \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}$$

HALF ANGLE FORMULAS

$$\text{With } s = \frac{a+b+c}{2}:$$

$$\text{(i)} \quad \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\text{(ii)} \quad \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$(iii) \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}$$

AREA

$$(i) \Delta = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B$$

$$(ii) \Delta = \sqrt{s(s-a)(s-b)(s-c)} \text{ (Heron)}$$

$$(iii) \Delta = \frac{abc}{4R} = rs = 2R^2 \sin A \sin B \sin C$$

CIRCLES ATTACHED TO THE TRIANGLE

$$(i) R = \frac{abc}{4\Delta} = \frac{a}{2 \sin A}$$

$$(ii) r = \frac{\Delta}{s} = (s-a) \tan \frac{A}{2} = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$(iii) r_1 = \frac{\Delta}{s-a} = s \tan \frac{A}{2} = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

$$(iv) r_1 + r_2 + r_3 - r = 4R$$

$$(v) \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

$$(vi) rr_1r_2r_3 = \Delta^2, \quad r_1r_2 + r_2r_3 + r_3r_1 = s^2$$

WHERE MARKS GO · WHICH $S-A$ GOES WITH WHICH

The excircle opposite A carries $s-a$ in the denominator and cosines at B and C . Mixing r_1 with $s-b$ is the standard error, and every identity that follows then fails.

PRODUCTS THAT COLLAPSE

$$(i) \cos \theta \cos 2\theta \cdots \cos 2^{n-1}\theta = \frac{\sin 2^n \theta}{2^n \sin \theta}$$

$$(ii) \sin \theta \sin(60^\circ - \theta) \sin(60^\circ + \theta) = \frac{1}{4} \sin 3\theta$$

$$(iii) \cos \theta \cos(60^\circ - \theta) \cos(60^\circ + \theta) = \frac{1}{4} \cos 3\theta$$

$$(iv) \tan \theta \tan(60^\circ - \theta) \tan(60^\circ + \theta) = \tan 3\theta$$

CONDITIONAL IDENTITIES, $A+B+C=\pi$

$$(i) \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$$

$$(ii) \cos 2A + \cos 2B + \cos 2C \\ = -1 - 4 \cos A \cos B \cos C$$

$$(iii) \tan A + \tan B + \tan C = \tan A \tan B \tan C$$

$$(iv) \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$$

$$(v) \Sigma \tan \frac{A}{2} \tan \frac{B}{2} = 1$$

CEVIANS AND CENTRES

$$(i) \text{Median from } A: m_a = \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2}$$

$$(ii) m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$$

$$(iii) \text{Angle bisector from } A: \frac{2bc \cos \frac{A}{2}}{b+c}$$

$$(iv) \text{Altitude from } A: \frac{2\Delta}{a}$$

$$(v) \text{Distance from the circumcentre to a side: } R \cos A$$

$$(vi) OI^2 = R^2 - 2Rr, \text{ with } O \text{ the circumcentre and } I \text{ the incentre}$$

REGULAR POLYGON OF N SIDES, SIDE A

$$(i) \text{Interior angle } \frac{(n-2)\pi}{n}, \text{ exterior } \frac{2\pi}{n}$$

$$(ii) \text{Area } \frac{na^2}{4} \cot \frac{\pi}{n}$$

$$(iii) \text{Circumradius } \frac{a}{2} \operatorname{cosec} \frac{\pi}{n}, \text{ inradius } \frac{a}{2} \cot \frac{\pi}{n}$$

HEIGHTS AND DISTANCES

(i) Elevation and depression are measured from the horizontal, and are equal for the same pair of points

(ii) The $m-n$ rule: $(m+n) \cot \theta = m \cot \alpha - n \cot \beta$

(iii) Equivalently $(m+n) \cot \theta = n \cot B - m \cot C$

SOLUTION OF TRIANGLES

GIVEN	USE
Three sides	cosine rule, then sine rule
Two sides, included angle	cosine rule for the third
Two angles and a side	sine rule
Two sides, non-included angle	ambiguous case, check for two triangles

CHAPTER 11 INVERSE TRIGONOMETRIC FUNCTIONS

PRINCIPAL DOMAIN AND RANGE

Every identity in this chapter is true only on the range printed here. Almost every mark lost in inverse trigonometry is a range error, not an algebra error.

	DOMAIN	PRINCIPAL RANGE
$\sin^{-1} x$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$\mathbb{R}$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
$\cot^{-1} x$	$\mathbb{R}$	$(0, \pi)$
$\sec^{-1} x$	$\mathbb{R} - (-1, 1)$	$[0, \pi] - \{\frac{\pi}{2}\}$
$\operatorname{cosec}^{-1} x$	$\mathbb{R} - (-1, 1)$	$[-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$

Three ranges sit in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and three in $[0, \pi]$. That split explains every sign rule in this chapter, so learn the split rather than the rules.

MONOTONICITY AND SHAPE

FUNCTION	BEHAVIOUR	EDGE
$\sin^{-1} x$	increasing	$\pm \frac{\pi}{2}$ at $x = \pm 1$
$\cos^{-1} x$	decreasing	π at -1 , 0 at 1
$\tan^{-1} x$	increasing	$y = \pm \frac{\pi}{2}$
$\cot^{-1} x$	decreasing	$y = \pi$ and $y = 0$
$\sec^{-1} x$	up on each branch	$y = \frac{\pi}{2}$
$\operatorname{cosec}^{-1} x$	down on each branch	$y = 0$

$\cot^{-1} x$ is continuous on all of $\mathbb{R}$ and never takes the value 0 or π . It does not jump at $x = 0$, unlike $\tan^{-1}(1/x)$.

VALUES TO RECOGNISE INSTANTLY

x	$\sin^{-1}$	$\cos^{-1}$	$\tan^{-1}$
-1	$-\frac{\pi}{2}$	π	$-\frac{\pi}{4}$
$-\frac{\sqrt{3}}{2}$	$-\frac{\pi}{3}$	$\frac{5\pi}{6}$	
$-\frac{1}{2}$	$-\frac{\pi}{6}$	$\frac{2\pi}{3}$	

0	0	$\frac{\pi}{2}$	0
$\frac{1}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	
$\frac{1}{\sqrt{3}}$			$\frac{\pi}{6}$
$\frac{1}{\sqrt{2}}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{6}$	
1	$\frac{\pi}{2}$	0	$\frac{\pi}{4}$
$\sqrt{3}$			$\frac{\pi}{3}$

CANCELLING: THE OUTER FORM

These hold on the whole domain, with no condition beyond the domain itself.

- (i) $\sin(\sin^{-1} x) = x$, $\cos(\cos^{-1} x) = x$, $x \in [-1, 1]$
- (ii) $\tan(\tan^{-1} x) = x$, $\cot(\cot^{-1} x) = x$, $x \in \mathbb{R}$
- (iii) $\sec(\sec^{-1} x) = x$, $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$, $|x| \geq 1$

CANCELLING: THE INNER FORM

These hold only on the principal range. Outside it the angle must be folded back in.

- (i) $\sin^{-1}(\sin x) = x$ only on $[-\frac{\pi}{2}, \frac{\pi}{2}]$
- (ii) $\cos^{-1}(\cos x) = x$ only on $[0, \pi]$
- (iii) $\tan^{-1}(\tan x) = x$ only on $(-\frac{\pi}{2}, \frac{\pi}{2})$
- (iv) $\cot^{-1}(\cot x) = x$ only on $(0, \pi)$

WHERE MARKS GO · THE ASYMMETRY TESTED

The outer form cancels everywhere on the domain; the inner form cancels only on the range. So $\sin^{-1}(\sin 3) = \pi - 3$, not 3 , because 3 lies outside $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

THE PIECEWISE RULE · FOLDING SINE BACK IN

$$\sin^{-1}(\sin x) = \begin{cases} x, & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \leq x \leq \frac{3\pi}{2} \\ x - 2\pi, & \frac{3\pi}{2} \leq x \leq \frac{5\pi}{2} \end{cases}$$

FOLDING THE OTHERS BACK IN

- (i) $\cos^{-1}(\cos x) = x$ on $[0, \pi]$, $= 2\pi - x$ on $[\pi, 2\pi]$, $= x - 2\pi$ on $[2\pi, 3\pi]$

- (ii) $\tan^{-1}(\tan x) = x - n\pi$, with n chosen so the result lies in $(-\frac{\pi}{2}, \frac{\pi}{2})$
- (iii) $\cot^{-1}(\cot x) = x - n\pi$, with the result in $(0, \pi)$
- (iv) Both piecewise graphs have period 2π ; $\cos^{-1}(\cos x)$ is even and $\sin^{-1}(\sin x)$ is odd
- (v) $\sin^{-1}(\sin x)$ is a triangular wave of amplitude $\frac{\pi}{2}$; $\cos^{-1}(\cos x)$ one of amplitude π sitting above the axis

Method: reduce x modulo 2π , then reflect with $\sin(\pi - x) = \sin x$ or $\cos(2\pi - x) = \cos x$ until the angle lands in the principal range.

NEGATIVE ARGUMENT

- (i) $\sin^{-1}(-x) = -\sin^{-1} x$
- (ii) $\tan^{-1}(-x) = -\tan^{-1} x$
- (iii) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$
- (iv) $\cos^{-1}(-x) = \pi - \cos^{-1} x$
- (v) $\cot^{-1}(-x) = \pi - \cot^{-1} x$
- (vi) $\sec^{-1}(-x) = \pi - \sec^{-1} x$

The three odd ones are exactly those whose range is symmetric about the origin. That is the pattern, not a list to memorise.

RECIPROCAL ARGUMENT

- (i) $\sin^{-1} \frac{1}{x} = \operatorname{cosec}^{-1} x$, $|x| \geq 1$
- (ii) $\cos^{-1} \frac{1}{x} = \sec^{-1} x$, $|x| \geq 1$
- (iii) $\tan^{-1} \frac{1}{x} = \cot^{-1} x$ for $x > 0$
- (iv) $\tan^{-1} \frac{1}{x} = \cot^{-1} x - \pi$ for $x < 0$

WHERE MARKS GO · THE $X < 0$ RECIPROCAL

$\tan^{-1}(1/x)$ is negative for $x < 0$ while $\cot^{-1} x$ is close to π , so the two differ by π there. Treating them as equal for all x is the most expensive habit in this chapter.

ALL THREE EQUAL $\pi/2$ · COMPLEMENTARY PAIRS

$$\begin{aligned}\sin^{-1} x + \cos^{-1} x &= \frac{\pi}{2}, \quad x \in [-1, 1] \\ \tan^{-1} x + \cot^{-1} x &= \frac{\pi}{2}, \quad x \in \mathbb{R} \\ \sec^{-1} x + \operatorname{cosec}^{-1} x &= \frac{\pi}{2}, \quad |x| \geq 1\end{aligned}$$

CONVERTING BETWEEN THE SIX

For $0 < x < 1$, reading a right triangle with the angle at the base:

- (i) $\sin^{-1} x = \cos^{-1} \sqrt{1-x^2} = \tan^{-1} \frac{x}{\sqrt{1-x^2}}$
- (ii) $\cos^{-1} x = \sin^{-1} \sqrt{1-x^2} = \tan^{-1} \frac{\sqrt{1-x^2}}{x}$
- (iii) $\tan^{-1} x = \sin^{-1} \frac{x}{\sqrt{1+x^2}} = \cos^{-1} \frac{1}{\sqrt{1+x^2}}$
- (iv) $\cot^{-1} x = \cos^{-1} \frac{x}{\sqrt{1+x^2}}$ for $x > 0$
- (v) $\sec^{-1} x = \cos^{-1} \frac{1}{x} = \tan^{-1} \sqrt{x^2-1}$ for $x > 1$

Each conversion assumes a first-quadrant angle. For negative x , convert the modulus and restore the sign with the negative-argument rules above.

EVERY YEAR · SUM OF TWO TANGENTS

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$$

valid when $xy < 1$

THE CORRECTION TERMS

- (i) $= \pi + \tan^{-1} \frac{x+y}{1-xy}$ when $x > 0, y > 0, xy > 1$
- (ii) $= -\pi + \tan^{-1} \frac{x+y}{1-xy}$ when $x < 0, y < 0, xy > 1$
- (iii) $= \frac{\pi}{2}$ when $x > 0$ and $xy = 1$; $= -\frac{\pi}{2}$ when $x < 0$ and $xy = 1$
- (iv) $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}$, valid when $xy > -1$

WHERE MARKS GO · WHY THE PAPER PICKS $X = Y = 2$

Then $xy = 4 > 1$ with both positive, so the answer is $\pi + \tan^{-1} \left(-\frac{4}{3}\right) = \pi - \tan^{-1} \frac{4}{3}$, not $\tan^{-1} \left(-\frac{4}{3}\right)$.

SUM AND DIFFERENCE: SINE AND COSINE

- (i) $\sin^{-1} x + \sin^{-1} y = \sin^{-1} (x\sqrt{1-y^2} + y\sqrt{1-x^2})$, valid when $x^2 + y^2 \leq 1$, or when $xy < 0$
- (ii) $= \pi - \sin^{-1}(\dots)$ when $x, y > 0$ and $x^2 + y^2 > 1$
- (iii) $= -\pi - \sin^{-1}(\dots)$ when $x, y < 0$ and $x^2 + y^2 > 1$
- (iv) $\sin^{-1} x - \sin^{-1} y = \sin^{-1} (x\sqrt{1-y^2} - y\sqrt{1-x^2})$
- (v) $\cos^{-1} x \pm \cos^{-1} y$

$$= \cos^{-1}(xy \mp \sqrt{1-x^2}\sqrt{1-y^2})$$

$$(vi) \cos^{-1}x + \cos^{-1}y = 2\pi - \cos^{-1}(\dots) \text{ when } x + y < 0$$

THREE TANGENTS

$$(i) \tan^{-1}x + \tan^{-1}y + \tan^{-1}z \\ = \tan^{-1} \frac{x+y+z-xyz}{1-xy-yz-zx}$$

(ii) The sum is π exactly when $x+y+z=xyz$, with all three positive

(iii) The sum is $\frac{\pi}{2}$ exactly when $xy+yz+zx=1$

$$(iv) \tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$$

$$(v) \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$$

CONDITIONS DO THE WORK · DOUBLE ANGLE

$$2 \tan^{-1}x = \tan^{-1} \frac{2x}{1-x^2}, |x| < 1 \\ = \sin^{-1} \frac{2x}{1+x^2}, |x| \leq 1 \\ = \cos^{-1} \frac{1-x^2}{1+x^2}, x \geq 0$$

DOUBLE AND TRIPLE ANGLE

$$(i) 2 \sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2}), |x| \leq \frac{1}{\sqrt{2}}$$

$$(ii) 2 \cos^{-1}x = \cos^{-1}(2x^2-1), x \geq 0$$

$$(iii) 3 \sin^{-1}x = \sin^{-1}(3x-4x^3), |x| \leq \frac{1}{2}$$

$$(iv) 3 \cos^{-1}x = \cos^{-1}(4x^3-3x), \frac{1}{2} \leq x \leq 1$$

$$(v) 3 \tan^{-1}x = \tan^{-1} \frac{3x-x^3}{1-3x^2}, |x| < \frac{1}{\sqrt{3}}$$

OUTSIDE THE STATED RANGE

EXPRESSION	VALUE
$\sin^{-1} \frac{2x}{1+x^2}, x > 1$	$\pi - 2 \tan^{-1}x$
$\sin^{-1} \frac{2x}{1+x^2}, x < -1$	$-\pi - 2 \tan^{-1}x$
$\cos^{-1} \frac{1-x^2}{1+x^2}, x < 0$	$-2 \tan^{-1}x$
$\tan^{-1} \frac{2x}{1-x^2}, x > 1$	$2 \tan^{-1}x - \pi$
$\tan^{-1} \frac{2x}{1-x^2}, x < -1$	$2 \tan^{-1}x + \pi$
$2 \cos^{-1}x, x < 0$	$2\pi - \cos^{-1}(2x^2-1)$

These six lines are the whole content of a large family of questions. Each is obtained by folding $2 \tan^{-1}x$ or $2 \cos^{-1}x$ back into the principal range.

SUBSTITUTIONS THAT CLEAR THE SURD

EXPRESSION	PUT
$\sqrt{1-x^2}$	$x = \sin \theta \text{ or } \cos \theta$
$\sqrt{1+x^2}$	$x = \tan \theta \text{ or } \cot \theta$
$\sqrt{x^2-1}$	$x = \sec \theta \text{ or } \operatorname{cosec} \theta$
$\sqrt{\frac{a-x}{a+x}}$	$x = a \cos 2\theta$
$\sqrt{\frac{x}{a+x}}$	$x = a \tan^2 \theta$

STANDARD SIMPLIFICATIONS

$$(i) \tan^{-1} \frac{\sqrt{1+x^2}-1}{x} = \frac{1}{2} \tan^{-1}x$$

$$(ii) \tan^{-1} \frac{1-\cos x}{\sin x} = \frac{x}{2}$$

$$(iii) \tan^{-1} \frac{\cos x}{1+\sin x} = \frac{\pi}{4} - \frac{x}{2}$$

$$(iv) \sin^{-1} \sqrt{\frac{1-\cos x}{2}} = \frac{x}{2} \text{ for } 0 \leq x \leq \pi$$

$$(v) \tan^{-1} \sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{x}{2} \text{ for } 0 \leq x < \pi$$

$$(vi) \cos^{-1} \frac{1-x^2}{1+x^2} + \tan^{-1} \frac{2x}{1-x^2} = 4 \tan^{-1}x \text{ for } 0 < x < 1$$

TELESCOPING SERIES

$$(i) \tan^{-1} \frac{1}{1+n+n^2} = \tan^{-1}(n+1) - \tan^{-1}n$$

$$(ii) \tan^{-1} \frac{2}{n^2} = \tan^{-1}(n+1) - \tan^{-1}(n-1)$$

$$(iii) \sum_{n=1}^{\infty} \tan^{-1} \frac{1}{1+n+n^2} = \frac{\pi}{4}$$

(iv) The difference form $\tan^{-1} \frac{x-y}{1+xy}$ is the engine behind every such sum

SOLVING EQUATIONS

(i) Take $\sin$, $\cos$ or $\tan$ of both sides, then test every root against the original ranges

$$(ii) \sin^{-1}x + \sin^{-1}y = \frac{\pi}{2} \iff x^2 + y^2 = 1 \text{ with } x, y \geq 0$$

$$(iii) \cos^{-1}x + \cos^{-1}y = \pi \iff y = -x$$

$$(iv) \sin^{-1}x = \cos^{-1}x \iff x = \frac{1}{\sqrt{2}}$$

$$(v) \tan^{-1}x = \cot^{-1}x \iff x = 1$$

Taking a trigonometric function of both sides can create roots

the original ranges forbid. Substituting back is part of the method, not a check you may skip.

RANGES OF THE STANDARD COMBINATIONS

- (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, a constant on $[-1, 1]$
- (ii) $(\sin^{-1} x)^2 + (\cos^{-1} x)^2$ has range $\left[\frac{\pi^2}{8}, \frac{5\pi^2}{4}\right]$
- (iii) $\sin^{-1} x \cdot \cos^{-1} x$ has maximum $\frac{\pi^2}{16}$ at $x = \frac{1}{\sqrt{2}}$
- (iv) $\tan^{-1} x + \tan^{-1} \frac{1}{x} = \frac{\pi}{2}$ for $x > 0$, and $-\frac{\pi}{2}$ for $x < 0$

CALCULUS OF THE INVERSE FUNCTIONS

$$(i) \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

$$(ii) \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}, \quad \frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}$$

$$(iii) \frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$$

$$(iv) \int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1-x^2} + C$$

$$(v) \int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$$

WHERE MARKS GO · SIMPLIFY FIRST

Before differentiating a composite such as $\tan^{-1} \frac{2x}{1-x^2}$, fold it with the rules above. Differentiating the raw expression gives an answer that is right on one interval and wrong on the next.

CHAPTER 12 LIMITS, CONTINUITY AND DIFFERENTIABILITY

MEMORISE, X IN RADIANS · STANDARD LIMITS

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \quad \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

THE REST OF THE STANDARD LIST

- (i) $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1, \quad \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$
- (ii) $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$
- (iii) $\lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{x} = n$
- (iv) $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$
- (v) $\lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x} = \lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x} = 1$
- (vi) $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e, \quad \lim_{x \rightarrow 0} (1+x)^{1/x} = e$
- (vii) $\lim_{x \rightarrow \infty} \frac{\ln x}{x^k} = 0$ and $\lim_{x \rightarrow \infty} \frac{x^k}{e^x} = 0$ for $k > 0$

INDETERMINATE FORMS AND WHAT TO DO

FORM	MOVE
$\frac{0}{0}, \frac{\infty}{\infty}$	L'Hopital, or expand in series
$0 \cdot \infty$	write as $\frac{0}{0}$ or $\frac{\infty}{\infty}$
$\infty - \infty$	common denominator, or rationalise
1^∞	$\lim f^g = e^{\lim g(f-1)}$
$0^0, \infty^0$	take logarithms

L'Hopital applies only to $\frac{0}{0}$ and $\frac{\infty}{\infty}$, and only when the derivative limit exists. Differentiate numerator and denominator separately, never as a quotient.

EXPANSIONS USED NEAR ZERO

$$(i) \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$(ii) \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$(iii) \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$$

$$(iv) e^x = 1 + x + \frac{x^2}{2!} + \dots$$

$$(v) \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$(vi) (1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \dots$$

$$(vii) \sin^{-1} x = x + \frac{x^3}{6} + \dots, \quad \tan^{-1} x = x - \frac{x^3}{3} + \dots$$

WHERE MARKS GO · EXPAND FAR ENOUGH

Truncating a series one term too early is how a limit that should be $\frac{1}{12}$ comes out as 0. Keep terms until the lowest surviving power appears in both numerator and denominator.

SANDWICH, AND SUMS AS INTEGRALS

- (i) If $g \leq f \leq h$ and $g, h \rightarrow L$, then $f \rightarrow L$
- (ii) $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n f\left(\frac{r}{n}\right) = \int_0^1 f(x) dx$
- (iii) Read $\frac{r}{n} \rightarrow x$, $\frac{1}{n} \rightarrow dx$, and the limits from the extreme values of $\frac{r}{n}$

CONTINUITY

- (i) Continuous at a when $\lim_{x \rightarrow a^-} f = \lim_{x \rightarrow a^+} f = f(a)$
- (ii) Sums, products and compositions of continuous functions are continuous
- (iii) A quotient is continuous wherever the denominator is not zero
- (iv) Intermediate value: a continuous f on $[a, b]$ takes every value between $f(a)$ and $f(b)$

- (v) A continuous function on a closed bounded interval attains its bounds

DIFFERENTIABILITY

- (i) $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$, when both one-sided limits agree
- (ii) Differentiable implies continuous. The converse is false
- (iii) $|x|$ is continuous but not differentiable at 0; a corner kills the derivative
- (iv) $x^{1/3}$ has a vertical tangent at 0, so it is not differentiable there
- (v) $x^2 \sin \frac{1}{x}$ with $f(0) = 0$ is differentiable at 0, but f' is not continuous there

THE MEAN VALUE THEOREMS

- (i) Rolle: f continuous on $[a, b]$, differentiable on (a, b) , $f(a) = f(b) \Rightarrow f'(c) = 0$ for some c
- (ii) Lagrange: the same first two conditions give $f'(c) = \frac{f(b) - f(a)}{b - a}$
- (iii) Cauchy: $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$

CHAPTER 13 DIFFERENTIATION

THE STANDARD DERIVATIVES

F	F'
x^n	nx^{n-1}
e^x	e^x
a^x	$a^x \ln a$
$\ln x$	$1/x$
$\log_a x$	$\frac{1}{x \ln a}$

$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{ x \sqrt{x^2-1}}$

RULES

- (i) Product: $(uv)' = u'v + uv'$
- (ii) Quotient: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$
- (iii) Chain: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

(iv) Parametric: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

(v) Inverse: $\frac{dx}{dy} = \frac{1}{dy/dx}$

WHERE MARKS GO · THE SECOND DERIVATIVE

$\frac{d^2y}{dx^2} \neq \frac{d^2y/dt^2}{d^2x/dt^2}$. The correct route is $\frac{d}{dt} \left(\frac{dy}{dx} \right) \div \frac{dx}{dt}$.

LOGARITHMIC DIFFERENTIATION

(i) Use it for $y = [f(x)]^{g(x)}$ and for long products and quotients

(ii) $\frac{d}{dx} [f^g] = f^g \left[g' \ln f + \frac{g f'}{f} \right]$

(iii) $\frac{d}{dx} x^x = x^x (1 + \ln x)$

IMPLICIT AND HIGHER ORDER

(i) Differentiate every term, attaching $\frac{dy}{dx}$ to each y term, then solve

(ii) $\frac{d^n}{dx^n} \sin x = \sin \left(x + \frac{n\pi}{2} \right)$

(iii) $\frac{d^n}{dx^n} e^{ax} = a^n e^{ax}$, $\frac{d^n}{dx^n} \ln x = \frac{(-1)^{n-1} (n-1)!}{x^n}$

(iv) Leibniz: $(uv)^{(n)} = \sum_{r=0}^n {}^nC_r u^{(n-r)} v^{(r)}$

DERIVATIVE OF A DETERMINANT

(i) Differentiate one row at a time and add the results

(ii) Three rows give three determinants, not one

SIMPLIFY BEFORE YOU DIFFERENTIATE

(i) $\tan^{-1} \frac{2x}{1-x^2} \rightarrow 2 \tan^{-1} x$ on $|x| < 1$, derivative $\frac{2}{1+x^2}$

(ii) $\sin^{-1} \frac{2x}{1+x^2} \rightarrow 2 \tan^{-1} x$ on $|x| \leq 1$

(iii) $\tan^{-1} \sqrt{\frac{1-\cos x}{1+\cos x}} \rightarrow \frac{x}{2}$

The derivative of the raw form is right on one interval and wrong on the next. Fold first, then differentiate.

DIFFERENTIALS AND APPROXIMATION

(i) $\Delta y \approx f'(x) \Delta x$

(ii) The percentage error in $y = x^n$ is n times that in x

(iii) $f(a+h) \approx f(a) + hf'(a)$

CHAPTER 14 APPLICATION OF DERIVATIVES

AT (X_1, Y_1) · TANGENT AND NORMAL

Tangent: $y - y_1 = m(x - x_1)$

Normal: $y - y_1 = -\frac{1}{m}(x - x_1)$

LENGTHS AT A POINT

(i) Tangent horizontal when $f' = 0$, vertical when f' is undefined

(ii) Length of tangent $\left| \frac{y_1 \sqrt{1+m^2}}{m} \right|$, subtangent $\left| \frac{y_1}{m} \right|$

(iii) Length of normal $|y_1 \sqrt{1+m^2}|$, subnormal $|y_1 m|$

(iv) Angle between two curves: $\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$

(v) Orthogonal curves: $m_1 m_2 = -1$

MONOTONICITY

(i) $f' > 0$ on an interval gives strict increase there

(ii) $f' < 0$ gives strict decrease

(iii) $f' \geq 0$ with equality only at isolated points still gives strict increase

(iv) To prove $g(x) > 0$ on $[a, \infty)$, show $g(a) \geq 0$ and $g' > 0$

MAXIMA AND MINIMA

(i) Critical points: $f' = 0$ or f' undefined

(ii) First derivative test: a sign change $+$ to $-$ gives a maximum, $-$ to $+$ a minimum

(iii) Second derivative test: $f'(c) = 0$ with $f''(c) < 0$ gives a maximum, $f''(c) > 0$ a minimum

(iv) If $f''(c) = 0$, go to the next non-vanishing derivative: even order gives an extremum, odd order a point of inflection

- (v) On a closed interval, compare all critical values with the two endpoint values

WHERE MARKS GO · ENDPOINTS AND CORNERS

A global extremum on $[a, b]$ can sit at an endpoint or at a corner, where f' never equals zero. Solving $f' = 0$ alone finds neither.

CONCAVITY AND INFLECTION

- (i) $f'' > 0$: concave up. $f'' < 0$: concave down
 (ii) Inflection where f'' changes sign, not merely where $f'' = 0$
 (iii) For a concave-up f : $f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}$

STANDARD OPTIMISATION RESULTS

- (i) Largest rectangle in a circle of radius r is the square of side $r\sqrt{2}$
 (ii) Largest rectangle under $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has area $2ab$
 (iii) Cone of greatest volume in a sphere of radius R :

height $\frac{4R}{3}$

- (iv) Cylinder of greatest volume in a sphere of radius R : height $\frac{2R}{\sqrt{3}}$
 (v) Of all triangles with a given perimeter, the equilateral has the greatest area

RATE OF CHANGE

- (i) $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$, the chain rule read as rates
 (ii) Sphere: $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$
 (iii) Cone: $V = \frac{1}{3}\pi r^2 h$, curved surface $\pi r \ell$
 (iv) Write the relation first, differentiate second, substitute the instant last

ROOTS AND GRAPHS

- (i) Between two roots of f lies a root of f'
 (ii) Between two roots of f' lies at most one root of f
 (iii) To count real roots of $f(x) = k$, find the local extrema and compare their values with k

CHAPTER 15 INDEFINITE INTEGRATION

STANDARD INTEGRALS

F	$\int F DX$
x^n	$\frac{x^{n+1}}{n+1}, n \neq -1$
$\frac{1}{x}$	$\ln x $
e^x	e^x
a^x	$\frac{a^x}{\ln a}$
$\sin x$	$-\cos x$
$\tan x$	$\ln \sec x $
$\cot x$	$\ln \sin x $
$\sec x$	$\ln \sec x + \tan x $
$\operatorname{cosec} x$	$\ln \operatorname{cosec} x - \cot x $
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$

THE SIX THAT NEED A FORM

(i) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$

(ii) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right|$

(iii) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right|$

(iv) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$

(v) $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln \left| x + \sqrt{x^2 \pm a^2} \right|$

(vi) $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a}$

SURDS THAT COME OUT IN TWO PIECES

(i) $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$

(ii) $\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln \left| x + \sqrt{x^2 + a^2} \right|$

(iii) $\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln \left| x + \sqrt{x^2 - a^2} \right|$

ORDER OF CHOICE: ILATE · BY PARTS

$$\int uv dx = u \int v dx - \int (u' \int v dx) dx$$

USING BY PARTS WELL

- (i) Inverse, Logarithmic, Algebraic, Trigonometric, Exponential: the earlier one is u
- (ii) $\int e^x [f(x) + f'(x)] dx = e^x f(x) + C$
- (iii) $\int e^{ax} \sin bx dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$
- (iv) $\int [f(x) + xf'(x)] dx = xf(x) + C$

RATIONAL FUNCTIONS

- (i) Reduce to a proper fraction first, then split into partial fractions
- (ii) $\frac{px+q}{ax^2+bx+c}$: write $px+q = \lambda \frac{d}{dx}(ax^2+bx+c) + \mu$
- (iii) $\frac{1}{ax^2+bx+c}$: complete the square, then use one of the six forms
- (iv) A repeated factor $(x-a)^2$ needs both $\frac{A}{x-a}$ and $\frac{B}{(x-a)^2}$
- (v) An irreducible quadratic factor needs $\frac{Ax+B}{x^2+px+q}$

TRIGONOMETRIC INTEGRALS

- (i) $\frac{1}{a+b \cos x}$ and its family: put $t = \tan \frac{x}{2}$
- (ii) $\frac{1}{a \sin^2 x + b \cos^2 x}$: divide by $\cos^2 x$, then put $t = \tan x$
- (iii) $\frac{a \sin x + b \cos x}{c \sin x + d \cos x}$: write the numerator as $\lambda(\text{denominator}) + \mu(\text{its derivative})$
- (iv) Odd power of $\sin$: put $t = \cos x$. Odd power of $\cos$: put $t = \sin x$
- (v) Both even: use the double angle formulas to drop the powers

WHERE MARKS GO · TWO ANSWERS CAN DIFFER

$\int 2 \sin x \cos x dx$ is $\sin^2 x + C$, $-\cos^2 x + C$ and $-\frac{1}{2} \cos 2x + C$ at once. They differ by a constant, so an option that does not match yours may still be the same antiderivative.

SUBSTITUTIONS WORTH KNOWING

INTEGRAND	PUT
surds in $(x-a)(x-b)$	$x = a \cos^2 \theta + b \sin^2 \theta$
$\frac{1}{(ax+b)\sqrt{cx+d}}$	$cx+d = t^2$
$\frac{1}{x^k \sqrt{ax^2+b}}$	$x = 1/t$
$\frac{x^2+1}{x^4+kx^2+1}$	divide by x^2 , put $t = x \mp \frac{1}{x}$

CHAPTER 16 DEFINITE INTEGRATION AND AREA

EVERY YEAR · THE KING PROPERTY

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

THE PROPERTIES THAT DO THE WORK

- (i) $\int_0^a f(x) dx = \int_0^a f(a-x) dx$
- (ii) $\int_{-a}^a f = 2 \int_0^a f$ if f is even, and 0 if f is odd
- (iii) $\int_0^{2a} f = 2 \int_0^a f$ if $f(2a-x) = f(x)$, and 0 if $f(2a-x) = -f(x)$
- (iv) $\int_0^{nT} f = n \int_0^T f$ for a function of period T

$$(v) \int_a^b f = \int_a^c f + \int_c^b f$$

RESULTS WORTH CARRYING

- (i) $\int_0^{\pi/2} \ln(\sin x) dx = -\frac{\pi}{2} \ln 2$
- (ii) $\int_0^{\pi} xf(\sin x) dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx$
- (iii) $\int_0^{\pi/2} \frac{\sin^n x dx}{\sin^n x + \cos^n x} = \frac{\pi}{4}$
- (iv) Wallis: $\int_0^{\pi/2} \sin^m x \cos^n x dx$
 $= \frac{(m-1)(m-3) \cdots (n-1)(n-3) \cdots}{(m+n)(m+n-2) \cdots} \times k$

Each product descends in steps of two and stops at 2 or 1, and an empty product counts as 1. Here $k = \frac{\pi}{2}$ when m and

n are both even, and $k = 1$ otherwise.

DIFFERENTIATING AN INTEGRAL · LEIBNIZ

$$\frac{d}{dx} \int_{u(x)}^{v(x)} f(t) dt = f(v)v' - f(u)u'$$

The variable of integration is a dummy; only the limits and any x inside f matter. If x sits inside the integrand as well, split it out first.

ESTIMATION AND INEQUALITIES

- (i) $m(b-a) \leq \int_a^b f \leq M(b-a)$, with m, M the bounds of f
- (ii) $f \leq g$ on $[a, b]$ gives $\int_a^b f \leq \int_a^b g$
- (iii) $\left| \int_a^b f \right| \leq \int_a^b |f|$
- (iv) Mean value: $\int_a^b f = (b-a)f(c)$ for some $c \in (a, b)$

WHERE MARKS GO · A BREAK INSIDE THE LIMITS

Applying the fundamental theorem across a point where the integrand is undefined gives a wrong, often negative, answer for a positive area. Split at every such point and check each piece.

GAMMA AND REDUCTION

- (i) $\int_0^\infty x^n e^{-x} dx = n!$ for a non-negative integer n
- (ii) $I_n = \int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$
- (iii) $I_n = \int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n} I_{n-2}$

AREA UNDER CURVES

- (i) $A = \int_a^b |f(x)| dx$ between the curve and the x -axis
- (ii) Between two curves: $\int_a^b [f_{\text{upper}} - f_{\text{lower}}] dx$
- (iii) With respect to y : $\int_c^d [x_{\text{right}} - x_{\text{left}}] dy$
- (iv) Find the intersection points before integrating; they are the limits

STANDARD AREAS

- (i) Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$: area πab
- (ii) Parabola $y^2 = 4ax$ and its latus rectum: $\frac{8a^2}{3}$
- (iii) Parabola $y^2 = 4ax$ and the line $y = mx$: $\frac{8a^2}{3m^3}$
- (iv) $y = x^2$ and $y^2 = x$: area $\frac{1}{3}$
- (v) Asteroid $x^{2/3} + y^{2/3} = a^{2/3}$: area $\frac{3\pi a^2}{8}$

CHAPTER 17 DIFFERENTIAL EQUATIONS

ORDER AND DEGREE

- (i) Order: the highest derivative present
- (ii) Degree: the power of that highest derivative, once the equation is free of radicals and fractions in the derivatives
- (iii) An equation containing $\sin y'$ or $e^{y'}$ has no degree
- (iv) The general solution of an n th order equation carries n arbitrary constants

VARIABLES SEPARABLE

- (i) $\frac{dy}{dx} = f(x)g(y) \Rightarrow \int \frac{dy}{g(y)} = \int f(x) dx$
- (ii) $\frac{dy}{dx} = f(ax+by+c)$: put $t = ax+by+c$

HOMOGENEOUS

- (i) $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$: put $y = vx$, so $\frac{dy}{dx} = v + x \frac{dv}{dx}$
- (ii) The equation then separates in v and x
- (iii) $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$: shift the origin to the intersection of the two lines. If they are parallel, put $t = a_1x + b_1y$

EVERY YEAR · LINEAR FIRST ORDER

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\text{IF} = e^{\int P dx}, \quad y \cdot \text{IF} = \int Q \cdot \text{IF} dx + C$$

TWO VARIATIONS

- (i) Linear in x instead: $\frac{dx}{dy} + P(y)x = Q(y)$
- (ii) Bernoulli $\frac{dy}{dx} + Py = Qy^n$: divide by y^n and put $t = y^{1-n}$

WHERE MARKS GO · NO CONSTANT IN THE IF

Adding a constant inside $\int P dx$ multiplies the integrating factor by a number, which cancels but wastes time. Note also that $e^{\ln f} = f$ only when $f > 0$, so keep track of the sign on each interval.

EXACT EQUATIONS

- (i) $M dx + N dy = 0$ is exact when $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
- (ii) $x dy + y dx = d(xy)$
- (iii) $\frac{x dy - y dx}{x^2} = d\left(\frac{y}{x}\right)$
- (iv) $\frac{x dy - y dx}{x^2 + y^2} = d\left(\tan^{-1} \frac{y}{x}\right)$

$$(v) \frac{x dx + y dy}{x^2 + y^2} = d\left(\frac{1}{2} \ln(x^2 + y^2)\right)$$

FORMING THE EQUATION

- (i) Differentiate the given family as many times as there are constants, then eliminate them
- (ii) Orthogonal trajectories: replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ and solve again

APPLICATIONS

- (i) Growth and decay: $\frac{dN}{dt} = kN \Rightarrow N = N_0 e^{kt}$
- (ii) Newton cooling: $\frac{d\theta}{dt} = -k(\theta - \theta_0)$
- (iii) Mixing: the rate of change of the amount is the rate in minus the rate out
- (iv) Geometry: the subtangent, the subnormal and the length of the tangent each translate into a first order equation

CHAPTER 18 STRAIGHT LINES AND CIRCLES

POINTS

- (i) Distance $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- (ii) Section in the ratio $m : n$ internally: $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$; externally, replace n by $-n$
- (iii) Centroid $\left(\frac{\Sigma x}{3}, \frac{\Sigma y}{3}\right)$
- (iv) Incentre $\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$
- (v) Area $\frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$
- (vi) Orthocentre, centroid and circumcentre are collinear, with $HG : GO = 2 : 1$

FORMS OF THE LINE

- (i) Slope intercept: $y = mx + c$
- (ii) Point slope: $y - y_1 = m(x - x_1)$
- (iii) Two point: $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$
- (iv) Intercept: $\frac{x}{a} + \frac{y}{b} = 1$
- (v) Normal: $x \cos \alpha + y \sin \alpha = p$

$$(vi) \text{ Parametric: } \frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r$$

EVERY YEAR · DISTANCE FROM A POINT

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Parallel lines: $d = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$

ANGLE, FOOT AND IMAGE

- (i) $\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$
- (ii) Parallel: $m_1 = m_2$. Perpendicular: $m_1 m_2 = -1$
- (iii) Foot of the perpendicular from (x_1, y_1) to $ax + by + c = 0$:
 $\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\frac{ax_1 + by_1 + c}{a^2 + b^2}$
- (iv) Image: the same expression with the right side doubled

FAMILIES AND BISECTORS

- (i) Any line through the intersection of $L_1 = 0$ and

$$L_2 = 0 \text{ is } L_1 + \lambda L_2 = 0$$

(ii) Three lines are concurrent when the determinant of their coefficients is 0

(iii) Angle bisectors:
$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

(iv) Make $c_1, c_2 > 0$; then the + sign gives the bisector of the angle containing the origin

PAIR OF STRAIGHT LINES

(i) $ax^2 + 2hxy + by^2 = 0$ is a pair through the origin, with $m_1 + m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$

(ii) $\tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$

(iii) Perpendicular when $a + b = 0$; coincident when $h^2 = ab$

(iv) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is a pair when $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

STANDARD FORMS · THE CIRCLE

$$(x - h)^2 + (y - k)^2 = r^2$$

$$x^2 + y^2 + 2gx + 2fy + c = 0, \text{ centre } (-g, -f),$$

$$\text{radius } \sqrt{g^2 + f^2 - c}$$

WORKING WITH A CIRCLE

(i) Diameter form: $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

(ii) Parametric: $x = h + r \cos \theta, y = k + r \sin \theta$

(iii) With $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$, the point is outside, on, or inside as S_1 is positive, zero or negative

(iv) Length of tangent $\sqrt{S_1}$

TANGENT, NORMAL, CHORD

(i) Tangent at (x_1, y_1) : $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

(ii) $y = mx \pm r\sqrt{1 + m^2}$ touches $x^2 + y^2 = r^2$ for every m

(iii) Chord of contact from an external point: $T = 0$

(iv) Chord with midpoint (x_1, y_1) : $T = S_1$

(v) The normal at any point passes through the centre

WHERE MARKS GO · TANGENCY IS A DISTANCE

A line touches a circle exactly when the distance from the centre equals the radius. Setting the discriminant of the substituted quadratic to zero works too, but is slower and easier to sign-slip.

TWO CIRCLES

(i) Radical axis $S_1 - S_2 = 0$, always perpendicular to the line of centres

(ii) Orthogonal when $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

(iii) Touch externally when $d = r_1 + r_2$, internally when $d = |r_1 - r_2|$

(iv) The common chord of intersecting circles is the radical axis

(v) Family through the intersection: $S_1 + \lambda S_2 = 0$, $\lambda \neq -1$

CHAPTER 19 CONIC SECTIONS

THE THREE AT A GLANCE

	PARABOLA	ELLIPSE	HYPERBOLA
Standard	$y^2 = 4ax$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
e	1	< 1	> 1
Focus	$(a, 0)$	$(\pm ae, 0)$	$(\pm ae, 0)$
Directrix	$x = -a$	$x = \pm a/e$	$x = \pm a/e$
Latus rectum	$4a$	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Relation		$b^2 = a^2(1 - e^2)$	$b^2 = a^2(e^2 - 1)$

PARAMETRIC POINTS

(i) Parabola: $(at^2, 2at)$

(ii) Ellipse: $(a \cos \theta, b \sin \theta)$

(iii) Hyperbola: $(a \sec \theta, b \tan \theta)$

(iv) Rectangular hyperbola $xy = c^2$: $\left(ct, \frac{c}{t}\right)$

TANGENTS

(i) Parabola at (x_1, y_1) : $yy_1 = 2a(x + x_1)$; at t : $ty = x + at^2$

- (ii) Ellipse at (x_1, y_1) : $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$
 (iii) Hyperbola at (x_1, y_1) : $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$
 (iv) $y = mx + c$ touches: parabola if $c = \frac{a}{m}$; ellipse if $c^2 = a^2m^2 + b^2$; hyperbola if $c^2 = a^2m^2 - b^2$

NORMALS

- (i) Parabola at t : $y + tx = 2at + at^3$
 (ii) Ellipse at θ : $ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2$
 (iii) Hyperbola at θ : $ax \cos \theta + by \cot \theta = a^2 + b^2$
 (iv) Three normals can be drawn to a parabola from a point, and $t_1 + t_2 + t_3 = 0$

FOCAL PROPERTIES

- (i) Parabola: $t_1 t_2 = -1$ for a focal chord; focal distance $x_1 + a$
 (ii) The semi latus rectum is the harmonic mean of the two focal segments
 (iii) Ellipse: $SP + S'P = 2a$; focal distances $a \pm ex_1$
 (iv) Hyperbola: $|SP - S'P| = 2a$
 (v) Reflection: a ray from the focus of a parabola leaves parallel to the axis

DIRECTOR CIRCLE

- (i) The locus of the point from which the tangents are perpendicular
 (ii) Parabola: the directrix itself

- (iii) Ellipse: $x^2 + y^2 = a^2 + b^2$
 (iv) Hyperbola: $x^2 + y^2 = a^2 - b^2$, which is empty when $a < b$
 (v) Auxiliary circle of an ellipse: $x^2 + y^2 = a^2$

ASYMPTOTES AND THE RECTANGULAR HYPERBOLA

- (i) Asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $y = \pm \frac{b}{a}x$
 (ii) Rectangular when $a = b$, and then $e = \sqrt{2}$
 (iii) Conjugate hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, with $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$
 (iv) For $xy = c^2$, the tangent at t is $x + t^2y = 2ct$

WHERE MARKS GO · WHICH AXIS IS MAJOR

In $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the foci lie on the x -axis only when $a > b$. If $b > a$, every formula swaps and e comes from $a^2 = b^2(1 - e^2)$. Compare a and b before writing anything else.

GENERAL CONIC

- (i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is a parabola when $h^2 = ab$, an ellipse when $h^2 < ab$, a hyperbola when $h^2 > ab$
 (ii) It degenerates when $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$
 (iii) Chord of contact and midpoint chord: $T = 0$ and $T = S_1$, as for the circle

CHAPTER 20 VECTOR ALGEBRA

MEMORISE · DOT PRODUCT

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta = a_1b_1 + a_2b_2 + a_3b_3$$

WORKING WITH THE DOT PRODUCT

- (i) $\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0$
 (ii) $|\vec{a}|^2 = \vec{a} \cdot \vec{a}$
 (iii) Projection of $\vec{a}$ on $\vec{b}$ is $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$
 (iv) Projection vector: $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}$
 (v) $|\vec{a} \pm \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 \pm 2\vec{a} \cdot \vec{b}$

MEMORISE · CROSS PRODUCT

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin \theta \hat{n}$$

WORKING WITH THE CROSS PRODUCT

- (i) $\vec{a} \parallel \vec{b} \iff \vec{a} \times \vec{b} = \vec{0}$
 (ii) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
 (iii) Unit vector perpendicular to both: $\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$
 (iv) Area of a triangle $\frac{1}{2}|\vec{a} \times \vec{b}|$; of a parallelogram $|\vec{a} \times \vec{b}|$
 (v) Parallelogram from its diagonals: $\frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$

$$(vi) |\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$$

SCALAR TRIPLE PRODUCT

$$(i) [\vec{a} \vec{b} \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(ii) Cyclic shifts leave it unchanged; swapping two vectors changes the sign

$$(iii) \text{ Coplanar } \iff [\vec{a} \vec{b} \vec{c}] = 0$$

$$(iv) \text{ Volume of a parallelepiped } |[\vec{a} \vec{b} \vec{c}]|$$

$$(v) \text{ Volume of a tetrahedron } \frac{1}{6} |[\vec{a} \vec{b} \vec{c}]|$$

BAC MINUS CAB · VECTOR TRIPLE PRODUCT

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

TWO MORE PRODUCTS

$$(i) (\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$$

(ii) The cross product is not associative, so the brackets matter

$$(iii) (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$$

WHERE MARKS GO · WHICH VECTOR IS INSIDE

The result of $\vec{a} \times (\vec{b} \times \vec{c})$ lies in the plane of $\vec{b}$ and $\vec{c}$, and the outside vector $\vec{a}$ supplies both dot products. Getting the two terms the wrong way round is the standard error here.

POSITION VECTORS

$$(i) \text{ Section formula: } \vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n} \text{ internally}$$

$$(ii) \text{ Midpoint } \frac{\vec{a} + \vec{b}}{2}; \text{ centroid } \frac{\vec{a} + \vec{b} + \vec{c}}{3}$$

$$(iii) \text{ Collinear points: } \vec{AB} = \lambda \vec{AC}$$

$$(iv) \text{ Any } \vec{r} \text{ in the plane of } \vec{a} \text{ and } \vec{b}: \vec{r} = x\vec{a} + y\vec{b}$$

ANGLE AND DIRECTION

$$(i) \cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

$$(ii) \text{ Angle bisectors of } \vec{a} \text{ and } \vec{b}: \text{ along } \frac{\vec{a}}{|\vec{a}|} \pm \frac{\vec{b}}{|\vec{b}|}$$

(iii) $\vec{a}$ makes equal angles with the axes when its components are equal

CHAPTER 21 THREE DIMENSIONAL GEOMETRY

DIRECTION COSINES AND RATIOS

$$(i) l = \cos \alpha, m = \cos \beta, n = \cos \gamma, \text{ and } l^2 + m^2 + n^2 = 1$$

$$(ii) \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1, \text{ so } \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$$

$$(iii) \text{ From ratios } \langle a, b, c \rangle: l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \text{ and similarly for } m \text{ and } n$$

$$(iv) \text{ Through two points: ratios } \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

TWO FORMS OF THE SAME THING · THE LINE

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda$$

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

WORKING WITH THE LINE

$$(i) \text{ A general point is } (x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$$

$$(ii) \text{ Angle between two lines: } \cos \theta = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| |\vec{b}_2|}$$

$$(iii) \text{ Perpendicular when } a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

(iv) A zero denominator means that coordinate is constant, not that you may divide by zero

FOOT, IMAGE, DISTANCE

$$(i) \text{ Foot of the perpendicular from } P: \text{ take the general point } Q(\lambda), \text{ impose } \vec{PQ} \cdot \vec{b} = 0, \text{ and solve the single linear equation for } \lambda$$

$$(ii) \text{ Image of } P \text{ in the line: } P' = 2Q - P \text{ once } Q \text{ is known}$$

$$(iii) \text{ Distance from a point: } \frac{|\vec{AP} \times \vec{b}|}{|\vec{b}|}$$

SKEW LINES · SHORTEST DISTANCE

$$d = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$$

PARALLEL, INTERSECTING, COPLANAR

- (i) Parallel lines: $d = \frac{|(\vec{a}_2 - \vec{a}_1) \times \vec{b}|}{|\vec{b}|}$
- (ii) The lines meet exactly when $[\vec{a}_2 - \vec{a}_1 \ \vec{b}_1 \ \vec{b}_2] = 0$
- (iii) The same condition written as a determinant is the coplanarity test

WHERE MARKS GO · ALONG, NOT PERPENDICULAR

A distance measured *along* a given direction is not the perpendicular distance. Travel from the point along that direction, meet the target, then measure. Reading past this one word costs the whole question.

THE PLANE

- (i) General: $ax + by + cz + d = 0$, normal $\langle a, b, c \rangle$
- (ii) Vector: $\vec{r} \cdot \vec{n} = d$
- (iii) Normal form: $lx + my + nz = p$, with p the distance from the origin

(iv) Intercept form: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

(v) Through three points:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

ANGLES AND DISTANCES WITH PLANES

(i) Distance from a point: $\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$

(ii) Between two planes: $\cos \theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}$

(iii) Between a line and a plane: $\sin \theta = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}||\vec{n}|}$

(iv) A line lies in the plane when $\vec{b} \cdot \vec{n} = 0$ and its point satisfies the plane

(v) Family through the intersection of two planes: $P_1 + \lambda P_2 = 0$

Note the swap: two planes use cosine, a line with a plane uses sine, because the angle is measured to the plane and not to its normal.

CHAPTER 22 PROBABILITY

THE BASICS

- (i) $P(A) = \frac{\text{favourable}}{\text{total}}, \quad 0 \leq P(A) \leq 1$
- (ii) $P(A') = 1 - P(A)$
- (iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- (iv) $P(A \cup B \cup C) = \Sigma P(A) - \Sigma P(A \cap B) + P(A \cap B \cap C)$
- (v) Mutually exclusive: $P(A \cap B) = 0$
- (vi) Odds in favour $m : n$ means $P = \frac{m}{m+n}$

EVERY YEAR · CONDITIONAL PROBABILITY

$$P(A | B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

MULTIPLICATION AND INDEPENDENCE

- (i) $P(A \cap B) = P(B)P(A | B)$
- (ii) Independent: $P(A \cap B) = P(A)P(B)$
- (iii) Independence of three events needs all four conditions, not only the pairwise ones
- (iv) If A and B are independent, so are A' and B , and

 A' and B'

WHERE MARKS GO · INDEPENDENT, NOT EXCLUSIVE

Two events with non-zero probability cannot be both. Mutually exclusive means one rules the other out, which is the strongest possible dependence.

TOTAL PROBABILITY AND BAYES

(i) $P(A) = \sum_i P(E_i)P(A | E_i)$ over a partition $\{E_i\}$

(ii) $P(E_k | A) = \frac{P(E_k)P(A | E_k)}{\sum_i P(E_i)P(A | E_i)}$

(iii) Read the question for which event is given and which is asked; that decides numerator and denominator

N TRIALS, PROBABILITY P · BINOMIAL

$$P(X = r) = {}^n C_r p^r q^{n-r}, \quad q = 1 - p$$

READING A BINOMIAL

(i) Mean np , variance npq , so the variance is always

less than the mean

- (ii) Most likely number of successes: $(n+1)p$ rounded down, unless it is an integer, when there are two
- (iii) At least one success: $1 - q^n$

RANDOM VARIABLES

- (i) $E(X) = \sum x_i p_i$, $\text{Var}(X) = E(X^2) - [E(X)]^2$
- (ii) $E(aX + b) = aE(X) + b$, $\text{Var}(aX + b) = a^2 \text{Var}(X)$
- (iii) The standard deviation is the positive square root

of the variance

COUNTING-HEAVY STANDARDS

- (i) Two dice: $P(\text{sum} = 7) = \frac{1}{6}$, the most likely sum
- (ii) Birthday style questions: use $1 - P(\text{all different})$
- (iii) Cards: 52 cards, 13 of each suit, 12 face cards, 4 aces
- (iv) Drawing without replacement changes the denominator each time; with replacement it does not

CHAPTER 23 STATISTICS

MEASURES OF CENTRAL TENDENCY

- (i) Mean $\bar{x} = \frac{\sum x_i}{n}$; for a frequency table, $\frac{\sum f_i x_i}{\sum f_i}$
- (ii) Step deviation: $\bar{x} = A + h \cdot \frac{\sum f_i u_i}{\sum f_i}$, with $u_i = \frac{x_i - A}{h}$
- (iii) Median of grouped data: $\ell + \frac{\frac{N}{2} - C}{f} \times h$
- (iv) Mode of grouped data: $\ell + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times h$
- (v) Empirical relation: $\text{Mode} \approx 3 \text{Median} - 2 \text{Mean}$

EVERY YEAR · VARIANCE

$$\sigma^2 = \frac{\sum x_i^2}{n} - \bar{x}^2, \quad \sigma = \sqrt{\sigma^2}$$

WORKING WITH THE VARIANCE

- (i) Grouped: $\sigma^2 = \frac{\sum f_i x_i^2}{\sum f_i} - \left(\frac{\sum f_i x_i}{\sum f_i} \right)^2$
- (ii) Adding k to every observation leaves σ unchanged
- (iii) Multiplying every observation by k multiplies σ by $|k|$ and σ^2 by k^2
- (iv) Coefficient of variation $\frac{\sigma}{\bar{x}} \times 100$

STANDARD RESULTS

- (i) Variance of the first n natural numbers: $\frac{n^2 - 1}{12}$

- (ii) Mean of the first n natural numbers: $\frac{n+1}{2}$
- (iii) Variance of one throw of a fair die: $\frac{35}{12}$
- (iv) Mean deviation about the mean $\frac{\sum |x_i - \bar{x}|}{n}$; about the median it is least

WHERE MARKS GO · CORRECTING AN OBSERVATION

When one reading is replaced, correct $\sum x$ and $\sum x^2$ first, then recompute the mean and the variance. Adjusting the variance directly from the old value almost never works, because $\bar{x}$ has moved too.

COMBINING TWO GROUPS

- (i) Combined mean $\bar{x} = \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2}{n_1 + n_2}$
- (ii) Combined variance, with $d_i = \bar{x}_i - \bar{x}$:

$$\sigma^2 = \frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}$$

QUICK SANITY CHECKS

- (i) The variance is never negative, and is zero only when every observation is equal
- (ii) σ carries the same units as the data; σ^2 does not
- (iii) The mean is the only one of the three that every observation can move
- (iv) For a symmetric distribution the mean, median and mode coincide
- (v) A variance larger than the mean rules out a binomial model

[A NOTE BEFORE YOU CLOSE IT]

Three things worth remembering

A formula you cannot place is worse than one you never learned. Every line in this book belongs to a move: a substitution that clears a surd, a condition that decides a sign, a shortcut that saves ninety seconds. If you cannot say which move a formula serves, go back to the chapter rather than to this page.

Read the conditions, not only the formulas. Almost every mark lost in inverse trigonometry, in the sum of two arctangents, in the tangency of a line to a conic, is lost to a condition that was skipped, never to algebra that was too hard.

Revise this book three times. The first pass, a week out, is slow and complete. The second, two days out, is only the gold cards. The third, on the morning of the paper, is only the dark ones, because those are the mistakes you are about to make.

Finally, tell us when we are wrong. If a line here does not match what you derive, write to academics@iitianforum.com with the page and the formula. Corrections go into the next edition, and the student who finds one is credited in it.

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