

CONCEPT MAP

PROBABILITY

5 TOPICS • 32 CORE IDEAS • ONE SHEET



01
EXPERIMENT, SAMPLE SPACE, EVENT

i. A **random experiment** has more than one possible outcome and none of them can be predicted in advance.

ii. The **sample space** S is the set of all outcomes; $n(S)$ is its size.

iii. An **event** is any subset of S . It **occurs** when the outcome that happens belongs to it.

iv. A **simple event** is a single outcome; a **compound event** contains more than one.

v. $\emptyset$ is the impossible event and S is the sure event.

Two coins: $S = \{HH, HT, TH, TT\}$, $n(S) = 4$. "At least one head" is the event $\{HH, HT, TH\}$.

Write the sample space down. Almost every wrong answer in this chapter comes from a sample space that was assumed rather than listed. If it is small, list it; if it is large, count it with the methods of section 07.

02
TYPES OF EVENTS

• **Mutually exclusive:** $A \cap B = \emptyset$; they cannot both happen.

• **Exhaustive:** $A_1 \cup A_2 \cup \dots \cup A_n = S$; at least one must happen.

• **Equally likely:** no outcome is favoured over another.

• **Complementary:** A and A' ; mutually exclusive and exhaustive together.

• **Independent:** one happening does not change the chance of the other (section 12).

$A \cap B = \emptyset$, so $P(A \cup B) = P(A) + P(B)$

Exclusive is not independent. They are opposite ideas. If A and B are mutually exclusive with non-zero probabilities, then A happening makes B impossible, so they are as dependent as two events can be.

03
THE PROBABILITY OF AN EVENT

For a finite sample space of equally likely outcomes,

KEY

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{favourable}}{\text{total}}$$

THE AXIOMS

i. $0 \leq P(A) \leq 1$ for every event.

ii. $P(S) = 1$ and $P(\emptyset) = 0$.

iii. $P(A \cup B) = P(A) + P(B)$ whenever A and B are mutually exclusive.

iv. Hence $P(A') = 1 - P(A)$, which follows from the three above rather than being assumed.

$P(A) = 0$ **does not mean impossible.** On an infinite sample space an event can have probability zero without being empty — picking exactly 0.5 from $[0, 1]$, for instance. On a finite space in which every outcome has non-zero probability the two do coincide; it is the non-zero weights, not the finiteness, that force it.

04
ODDS

If the odds in favour of A are $a : b$, then

KEY

$$P(A) = \frac{a}{a+b}, \quad P(A') = \frac{b}{a+b}$$

and the odds against A are $b : a$. Conversely,

odds in favour = $\frac{P(A)}{1 - P(A)}$

Odds are a ratio, not a probability. Odds of 3 : 1 in favour mean $P = \frac{3}{4}$, not $\frac{3}{5}$. Read the word "against" carefully — it swaps the two numbers. Odds are undefined for a certain event, since $1 - P(A) = 0$.

05
ADDITION THEOREM

KEY

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

- Mutually exclusive: the last term is zero.
- Exactly one of A, B : $P(A) + P(B) - 2P(A \cap B)$.
- Neither: $P(A' \cap B') = 1 - P(A \cup B)$.
- A but not B : $P(A \cap B') = P(A) - P(A \cap B)$.

Two different "or"s. " A or B " in a question almost always means the inclusive union. "Exactly one of them" is the second bullet, and the two answers differ by $P(A \cap B)$.

06
THREE EVENTS

KEY

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &\quad - P(A \cap B) - P(B \cap C) \\ &\quad - P(C \cap A) + P(A \cap B \cap C) \end{aligned}$$

$P(A \cup B \cup C) = \Sigma_1 - \Sigma_2 + \Sigma_3$

Writing $\Sigma_1, \Sigma_2, \Sigma_3$ for the sums of the single, pairwise and triple probabilities, this is $\Sigma_1 - \Sigma_2 + \Sigma_3$, and

exactly one = $\Sigma_1 - 2\Sigma_2 + 3\Sigma_3$

Exactly two is $\Sigma_2 - 3\Sigma_3$, and **at least two** is $\Sigma_2 - 2\Sigma_3$. The same three sums answer every "how many of the three" question.

07
COUNTING THE SAMPLE SPACE

In JEE Advanced the sample space is rarely handed to you. Build it with the counting principles:

KEY

$$P = \frac{\text{favourable arrangements or selections}}{\text{total arrangements or selections}}$$

i. Decide first whether order matters: " n P_r " if it does, " n C_r " if it does not.

ii. Count the numerator and the denominator the **same** way; mixing the two is the usual error.

iii. For "at least one", count the complement and subtract.

iv. With repetition allowed, " n " replaces " r ".

Consistency beats cleverness. If the denominator treats the objects as distinguishable, the numerator must too. Two correct counts done under different conventions give a wrong ratio.

08
GEOMETRIC PROBABILITY

When the outcome is a point chosen at random from a region, probability becomes a ratio of measures:

KEY

$$P = \frac{\text{favourable length, area or volume}}{\text{total length, area or volume}}$$

$P = \frac{\text{favourable area}}{\text{total area}}$

The route for inequality questions. " x and y are chosen at random from $[0, 1]$; find $P(y > x)$ " is not a counting problem. Draw the square, shade the region the inequality describes, and divide the areas.

09
CONDITIONING & INDEPENDENCE

09
CONDITIONAL PROBABILITY

Given that B has happened, B becomes the new sample space:

KEY

$$P(A | B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

$P(A | B) = \Sigma_i P(E_i) P(A | E_i)$

Break the event over the causes and add. Each E_i is one route to A ; the term $P(E_i)P(A | E_i)$ is the chance of taking that route and arriving. The routes are disjoint, so the terms simply add.

10
PROPERTIES OF CONDITIONAL PROBABILITY

For a fixed B , the map $A \mapsto P(A | B)$ is itself a probability:

KEY

$$P(S | B) = 1, \quad P(A' | B) = 1 - P(A | B)$$

and the addition theorem carries over unchanged:

$$P((A \cup C) | B) = P(A | B) + P(C | B) - P(A \cap C | B)$$

Only the first slot behaves. These rules hold in A with B fixed. Nothing similar is true in the second slot: $P(A | B) + P(A | B')$ need not be 1. The two are linked, but through total probability (section 14), not by the complement rule.

11
MULTIPLICATION THEOREM

KEY

$$P(A \cap B) = P(B) P(A | B) = P(A) P(B | A)$$

CHAIN RULE

$$P(A \cap B \cap C) = P(A) P(B | A) P(C | A \cap B)$$

This is drawing without replacement. Three cards, all spades: $\frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50}$. Each factor is conditioned on everything already drawn, which is exactly the chain rule read left to right.

12
INDEPENDENT EVENTS

A and B are independent when knowing one tells you nothing about the other.

KEY

$$P(A \cap B) = P(A) P(B) \iff P(A | B) = P(A) \quad (P(B) > 0)$$

- If A and B are independent, so are A' and B ; A and B' ; A' and B' .
- P (at least one) = $1 - P(A')P(B')$.
- Independence is an assumption about the experiment, or a fact to be verified — never something to assume because the question is silent.

Test it, do not feel it. Independence has one test: multiply the two probabilities and compare with $P(A \cap B)$. Two events drawn from the same physical trial are often dependent even when they look unrelated.

13
PAIRWISE AND MUTUAL INDEPENDENCE

Three events are **mutually** independent if and only if all four conditions hold:

KEY

$$P(A \cap B) = P(A)P(B), \text{ and the other two pairs}$$
$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

Pairwise is weaker. All three pairs can be independent while the triple condition fails. Only mutual independence lets you multiply all three, so a question that says merely "pairwise independent" is warning you not to.

14
TOTAL PROBABILITY

If $E_1, E_2, \dots, E_n$ are mutually exclusive, exhaustive and each of non-zero probability, they **partition** S , and for any event A ,

KEY

$$P(A) = \sum_i P(E_i) P(A | E_i)$$

$P(A) = \Sigma_i P(E_i) P(A | E_i)$

15
BAYES' THEOREM

KEY

$$P(E_i | A) = \frac{P(E_i) P(A | E_i)}{\sum_j P(E_j) P(A | E_j)}$$

i. The denominator is exactly $P(A)$ from section 14.

ii. $P(E_i)$ are the **prior** probabilities, $P(E_i | A)$ the **posterior** ones.

iii. The posteriors add to 1 over all i — a free check on the arithmetic.

Bayes runs the tree backwards. Total probability goes cause $\rightarrow$ effect; Bayes is given the effect and asks which cause. Identify the partition first, then the common event, and the formula writes itself.

16
RANDOM VARIABLE AND ITS DISTRIBUTION

A random variable X attaches a number to every outcome. Its probability distribution lists the values with their probabilities:

KEY

$$\sum_i P(X = x_i) = 1, \quad P(X = x_i) \geq 0$$

X	VALUES
X	$x_1, x_2, \dots, x_n$
$P(X)$	$p_1, p_2, \dots, p_n$

Check the total before anything else. If the distribution carries an unknown k , the equation $\sum p_i = 1$ is what determines it, and it is also the check that a printed table is a genuine distribution.

17
MEAN OF A RANDOM VARIABLE

KEY

$$\mu = E(X) = \sum_i x_i p_i$$

- The mean need not be one of the values X can take.
- $E(X)$ is the long-run average value over many repetitions.
- $g(E(X)) = \sum g(x_i) p_i$ for any function g .

Also called the expectation. "Expected value", "mean" and $E(X)$ are three names for the same sum. A fair game is one whose expected gain is zero.

18
VARIANCE AND STANDARD DEVIATION

KEY

$$\begin{aligned} \sigma^2 &= \text{Var}(X) = \sum_i x_i^2 p_i - \mu^2 \\ &= E(X^2) - [E(X)]^2 \end{aligned}$$

and the standard deviation is $\sigma = \sqrt{\text{Var}(X)}$.

- $\text{Var}(X) \geq 0$, and it is zero only when X is constant.
- $E(X^2) \geq [E(X)]^2$ always, which is the same statement.
- The second form is nearly always the faster one to compute.

Square the values, not the probabilities. $E(X^2) = \sum x_i^2 p_i$. Writing $\sum x_i p_i^2$ is the single commonest slip in this section.

19
MEAN AND VARIANCE UNDER A CHANGE

KEY

$$\begin{aligned} E(aX + b) &= aE(X) + b \\ \text{Var}(aX + b) &= a^2 \text{Var}(X) \end{aligned}$$

- The shift b moves the mean but leaves the variance alone.
- $\sigma_{\text{new}} = |a| \sigma$ — the modulus matters when $a < 0$.
- $E(X + Y) = E(X) + E(Y)$ always; $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$ only when X and Y are independent.

Adding a constant changes nothing about spread. This is the same property that section 26 states for data, and it is a favourite: "every observation is increased by 5; find the new variance" has the answer "unchanged".

20
BERNOULLI TRIALS AND THE BINOMIAL

Bernoulli trials are finitely many, independent, with only two outcomes and the same p each time. Then for $r = 0, 1, \dots, n$,

KEY

$$P(X = r) = {}^n C_r p^r q^{n-r}, \quad q = 1 - p$$

$n = 6, p = \frac{1}{2}$; the peak sits at $r = np$

The four conditions all matter. Drawing with replacement is binomial; drawing without replacement is not, because the trials stop being independent and p stops being constant.

21
MEAN, VARIANCE AND MODE OF THE BINOMIAL

KEY

$$\mu = np, \quad \sigma^2 = npq, \quad \sigma = \sqrt{npq}$$

- Variance $<$ mean whenever $0 < p \leq 1$, since $q < 1$; equality only in the degenerate case $p = 0$.
- For $0 < p < 1$ the mode is $\lfloor (n+1)p \rfloor$; if $(n+1)p$ is a whole number there are two modes, $(n+1)p$ and $(n+1)p - 1$.
- Given μ and σ^2 , recover $q = \sigma^2/\mu$, then p and n .

"At least one" is the standard question. Never sum the terms; use the complement, $P(X \geq 1) = 1 - q^n$. "At most one" is $q^n + npq^{n-1}$.

22
POISSON DISTRIBUTION

When n is large, p small and $np = \lambda$ stays moderate, the binomial tends to

KEY

$$P(X = r) = \frac{e^{-\lambda} \lambda^r}{r!}, \quad r = 0, 1, 2, \dots$$

- Mean = variance = λ , which is the signature of a Poisson.
- $\sum_{r=0}^{\infty} P(X = r) = e^{-\lambda} e^{\lambda} = 1$.
- It models rare events in a fixed interval: misprints per page, calls per minute.

Advanced only, and rarely. The JEE Advanced syllabus lists Poisson for introductory properties at the top of the difficulty range. If a question gives mean = variance, it is pointing at a Poisson.

23
MEAN, MEDIAN AND MODE

KEY

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$

- Median:** the middle value once the data are ordered; for grouped data $\ell + \frac{N/2 - C}{f} \times h$.
- Mode:** the most frequent value; for grouped data $\ell + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times h$.
- The mean uses every observation, so a single extreme value moves it; the median and mode barely react.

Mode = 3 Median – 2 Mean

24
MEASURES OF DISPERSION

- Range:** largest – smallest. Uses only two observations.
- Quartile deviation:** $\frac{Q_3 - Q_1}{2}$.
- Mean deviation about the mean:** $\frac{\sum f_i |x_i - \bar{x}|}{\sum f_i}$.
- Mean deviation about the median:** the same with $\bar{x}$ replaced by the median.
- Variance and standard deviation:** section 25.

both have the same mean

small σ

large σ

$$\sigma^2 = \frac{\sum (x_i - \bar{x})^2}{n}$$

Mean deviation is least about the median. Among all choices of centre, the median minimises $\sum |x_i - a|$ while the mean minimises $\sum (x_i - a)^2$.

25
VARIANCE AND STANDARD DEVIATION OF DATA

KEY

$$\sigma^2 = \frac{\sum (x_i - \bar{x})^2}{n} = \frac{\sum x_i^2}{n} - \bar{x}^2$$

and for a frequency distribution with $N = \sum f_i$,

$$\sigma^2 = \frac{\sum f_i x_i^2}{N} - \left(\frac{\sum f_i x_i}{N} \right)^2$$

- Variance of the first n natural numbers: $\frac{n^2 - 1}{12}$.
- σ carries the same units as the data; σ^2 does not.
- $\sigma = 0$ exactly when every observation is equal.

26
CHANGE OF ORIGIN AND SCALE

If every observation x_i is replaced by $ax_i + b$:

KEY

$$\bar{x}_{\text{new}} = a\bar{x} + b, \quad \sigma_{\text{new}}^2 = a^2 \sigma^2, \quad \sigma_{\text{new}} = |a| \sigma$$

The constant b does not touch the spread. Change of origin shifts the mean and leaves variance and standard deviation exactly as they were. A question that adds a constant to every observation and asks for the new variance is asking you to notice this, and the answer is "unchanged".

27
COMBINED MEAN AND VARIANCE

For two groups of sizes n_1, n_2 with means $\bar{x}_1, \bar{x}_2$ and variances σ_1^2, σ_2^2 :

KEY

$$\bar{x} = \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2}{n_1 + n_2}$$

and with $d_1 = \bar{x}_1 - \bar{x}$, $d_2 = \bar{x}_2 - \bar{x}$,

$$\sigma^2 = \frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}$$

Variances do not simply average. The d^2 terms account for the two groups sitting at different centres, so the combined spread is larger than a weighted average of the two spreads unless the means coincide.

28
COEFFICIENT OF VARIATION

KEY

$$CV = \frac{\sigma}{|\bar{x}|} \times 100$$

- It is a pure number, so it compares the spread of data measured in different units.
- Smaller CV means more consistent, more uniform, more stable.
- It is undefined when $\bar{x} = 0$ and unreliable when $\bar{x}$ is near zero.

The word to look for is "consistent". "Which batsman is more consistent" is never answered by the standard deviation alone — it is answered by the coefficient of variation, because the two players have different averages.

29
METHOD & TRAPS

29 **WHICH TOOL THE QUESTION WANTS**

Almost every error here is a tool-selection error made in the first ten seconds. Run the question through this before writing anything.

WHAT THE QUESTION SAYS	WHAT IT WANTS
no condition, several routes	total probability
cause given, find the effect	multiplication theorem
effect given, find the cause	Bayes' theorem
both happen, nothing stated	test independence first
no condition, one route	favourable over total

A plausible answer is no evidence. Choosing Bayes when the question wanted total probability, or the reverse, still returns a number between 0 and 1, and papers are set so that it matches a printed option. Decide which quantity is being asked for before computing, never afterwards from the look of the answer.

30
THE STANDARD MODELS

EXPERIMENT	COUNTS
One die	$n(S) = 6$; prime $\{2, 3, 5\}$, so $P = \frac{1}{2}$
Two dice	$n(S) = 36$; sum 7 is the commonest, $P = \frac{1}{6}$
n coins	$n(S) = 2^n$; exactly r heads: " $C_r/2^n$ "
A pack of cards	52; 26 red, 13 per suit, 12 face, 4 aces
A leap year	52 weeks + 2 days, so P (53 Sundays) = $\frac{2}{7}$

Two dice are distinguishable. The sample space has 36 ordered pairs, not 21 unordered ones. Treating them as identical is what makes "sum = 7" come out wrong.

31
STANDARD RESULTS WORTH MEMORISING

i. $P(A') = 1 - P(A)$ and $P(A' \cap B') = 1 - P(A \cup B)$.

ii. Exactly one of A, B : $P(A) + P(B) - 2P(A \cap B)$.

iii. $P(A \cap B) = P(B)P(A | B) = P(A)P(B | A)$.

iv. At least one of n independent events: $1 - \prod (1 - p_i)$.

v. Binomial: $\mu = np$, $\sigma^2 = npq$, mode = $\lfloor (n+1)p \rfloor$.

vi. $\text{Var}(X) = E(X^2) - [E(X)]^2$.

vii. σ^2 of the first n naturals = $\frac{n^2 - 1}{12}$.

viii. Change of scale: $\bar{x} \rightarrow a\bar{x} + b$, $\sigma^2 \rightarrow a^2 \sigma^2$.

ix. Mode = 3 Median – 2 Mean, approximately.

32
QUICK FACTS AND COMMON TRAPS

i. Mutually exclusive and independent are not synonyms; for events of non-zero probability they exclude each other.

ii. Never assume independence because the question is silent — test it.

iii. $\text{Var}(X) = E(X^2) - \mu^2$: square the values, not the probabilities.

iv. Binomial needs replacement; without it the trials are not independent.

v. For a binomial with $0 < p \leq 1$ the variance is less than the mean.

vi. Adding a constant to every observation leaves the variance unchanged.

vii. $\sigma_{\text{new}} = |a| \sigma$ — keep the modulus when the scale is negative.

viii. Combined variance needs the d^2 terms; variances do not average.

ix. Consistency questions want the coefficient of variation, not σ .

x. Counting the sample space is Permutations and Combinations; that sheet has the tools.

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