

CONCEPT MAP

APPLICATION OF DERIVATIVES

5 TOPICS • 36 CORE IDEAS • ONE SHEET



A TANGENTS, NORMALS & RATES 01-09

81. THE DERIVATIVE AS A SLOPE AND AS A RATE A-01

At a point where f is differentiable, one number answers two questions:

$$f'(x_1) = \lim_{h \rightarrow 0} \frac{f(x_1 + h) - f(x_1)}{h}$$

- Geometrically it is the slope of the tangent to $y = f(x)$ at that point.
- Physically it is the instantaneous rate at which y changes with x .
- $f'(x_1) = \tan \theta$, where θ is the angle the tangent makes with the positive x -axis.
- If $|f'(x)| \rightarrow \infty$ as $x \rightarrow x_1$ the tangent is **vertical** — watch for $\sqrt{x}$ and other fractional powers.

Every idea on this sheet is one of those two readings. Tangents, rates, monotonicity and extrema are all questions about the sign or the size of f' ; concavity and the derivative tests are the same questions asked of f'' .

82. RATE OF CHANGE AND RELATED RATES A-02

If two quantities are linked and both vary with time, differentiate the link with respect to t .

KEY

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

- Name the variables and write the equation that ties them together.
- Differentiate that equation with respect to t , using the chain rule throughout.
- Substitute the instantaneous values after differentiating, never before.
- A quantity is increasing when its derivative is positive, decreasing when it is negative.

A sphere's radius grows at 2 cm/s. Since $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 8\pi r^2$.

Substituting too early is the standard error. Putting $r = 5$ into $V = \frac{4}{3}\pi r^3$ first makes V a constant and dV/dt zero. Differentiate the general relation, then substitute.

83. TANGENT AND NORMAL A-03

At $P(x_1, y_1)$ on $y = f(x)$, with $m = f'(x_1)$:

KEY

tangent : $y - y_1 = m(x - x_1)$

normal : $y - y_1 = -\frac{1}{m}(x - x_1)$

- The normal is perpendicular to the tangent at the same point, so their slopes multiply to -1 . If $m = 0$ the normal is the vertical line $x = x_1$.
- For a curve given implicitly, find dy/dx by implicit differentiation and evaluate at P .
- For a parametric curve, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

The point must lie on the curve. If the question gives a point that does not satisfy the equation, it is asking for a tangent from that point: put the point into the tangent at a general point $(t, f(t))$ and solve for t .

84. TANGENTS IN SPECIAL POSITIONS A-04

CONDITION	WHAT IT MEANS
$f'(x_1) = 0$	tangent parallel to the x -axis
$f'(x_1) = \infty$	tangent parallel to the y -axis
$f'(x_1) = 1$	tangent makes 45° with the x -axis
tangent $\parallel ax + by + c = 0$	$f'(x_1) = -a/b$
normal $\parallel ax + by + c = 0$	$f'(x_1) = b/a$

Tangent at the origin. If a curve passes through the origin, the tangent there is obtained by equating to zero the **lowest degree** terms of its equation. For $y^2 = x^2 + x^2$ the lowest terms are $y^2 - x^2 = 0$, giving $y = \pm x$.

85. ANGLE BETWEEN TWO CURVES A-05

At a common point, with tangent slopes m_1 and m_2 :

KEY

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

- Orthogonal curves:** $m_1 m_2 = -1$ at every point of intersection, or one tangent horizontal and the other vertical.
- Touching curves:** $m_1 = m_2$, so $\theta = 0$ and the curves share a tangent.
- The angle is defined only at a point the two curves actually share, so solve them simultaneously first.
- Find the intersection before the slopes.** Two curves may cross more than once and the angle can differ from point to point, so the answer belongs to a named point, not to the pair of curves.

86. TANGENT, NORMAL, SUB-TANGENT, SUB-NORMAL A-06

With $P(x_1, y_1)$ on the curve and $m = f'(x_1)$:

KEY

sub-tangent = $\left| \frac{y_1}{m} \right|$, sub-normal = $|y_1 m|$

$$PT = \left| \frac{y_1}{m} \right| \sqrt{1 + m^2}, \quad PN = |y_1| \sqrt{1 + m^2}$$

Two classical results. For $y^2 = 4ax$ the sub-normal is the constant $2a$; for $xy = c^2$ the sub-tangent at any point equals the modulus of the abscissa of that point. Both are one-line checks with the formulas above.

87. APPROXIMATION BY DIFFERENTIALS A-07

For a small change Δx near a point where f is differentiable,

KEY

$$f(x + \Delta x) \approx f(x) + f'(x) \Delta x$$

- Absolute error** in y is $\Delta y \approx f'(x) \Delta x$.
- Relative error** is $\frac{\Delta y}{y}$ and **percentage error** is $100 \frac{\Delta y}{y}$.
- Choose x to be the nearest point at which f is easy to evaluate exactly.

$\sqrt{25.3}$: take $f(x) = \sqrt{x}$, $x = 25$, $\Delta x = 0.3$. Then $\sqrt{25.3} \approx 5 + \frac{0.3}{10} = 5.03$.

The tangent line is the approximation. This formula is just the tangent at x used as a stand-in for the curve, which is why it is accurate only for small Δx and fails near a sharp bend.

88. ERRORS THAT PROPAGATE A-08

If a measured quantity carries a small percentage error, the quantities computed from it carry a multiple of it:

KEY

$$y = x^n \Rightarrow \frac{\Delta y}{y} \approx n \frac{\Delta x}{x}$$

- A 1% error in the radius gives about 2% in a surface area and 3% in a volume.
- For a product $y = uv$ the relative errors add; for a quotient $y = u/v$ the **signed** relative errors subtract, so the worst case is again the sum of their magnitudes.
- Errors are quoted as magnitudes, so take the modulus at the end.

Take logarithms first. Writing $\ln y = n \ln x$ and differentiating turns any power-product relation into a sum of relative errors in one step, which is faster than differentiating the original.

89. VERTICAL TANGENTS AND SHARP CORNERS A-09

A point can defeat the derivative in two different ways, and the paper tests both.

- Vertical tangent:** f is continuous at a and $|f'(x)| \rightarrow \infty$ as $x \rightarrow a$. The tangent is the line $x = a$. Watch for $\sqrt{x}$, $x^{2/3}$ and other fractional powers.
- Corner:** the one-sided derivatives exist and differ, so no single tangent line exists.
- Both are candidates for an extremum, because neither needs $f'(a) = 0$ to be one.

THE SHARP-CORNER RULE FOR $|f(x)|$

- If $f(a) = 0$ and $f'(a) \neq 0$, then $|f|$ is not differentiable at a .
- If $f(a) = 0$ and $f'(a) = 0$, then $|f|$ is differentiable at a .
- Away from the zeros of f , $|f|$ inherits the differentiability of f .

Count the zeros, then test each. "How many points of non-differentiability does $|f(x)|$ have?" is answered by listing the roots of f and checking f' at each, not by drawing the graph.

B MEAN VALUE THEOREMS 10-14

90. ROLLE'S THEOREM B-10

If f is continuous on $[a, b]$, differentiable on (a, b) , and $f(a) = f(b)$, then there is at least one $c \in (a, b)$ with

KEY

$$f'(c) = 0$$

Look for the sign change. $f(x) = |x|$ on $[-1, 1]$ has equal ends but no interior zero of f' .

- The theorem asserts existence, not uniqueness or a location.
- Geometrically: a curve that returns to its starting height must level off somewhere in between.

Between two roots of f lies a root of f' . This is Rolle applied with $f(a) = f(b) = 0$, and it is the engine of every root-counting argument in section 14.

91. LAGRANGE'S MEAN VALUE THEOREM B-11

If f is continuous on $[a, b]$ and differentiable on (a, b) , there is a $c \in (a, b)$ with

KEY

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Some tangent is parallel to the chord joining the end points.

- Rolle is the special case $f(a) = f(b)$.
- Rearranged: $f(b) - f(a) = f'(c)(b - a)$, which is the form used in every proof below.

This one theorem proves the derivative test. The sign of $f(b) - f(a)$ is the sign of $f'(c)$, because $b - a > 0$. Everything section 16 says about monotonicity is that sentence.

92. CAUCHY'S MEAN VALUE THEOREM B-12

If f, g are continuous on $[a, b]$, differentiable on (a, b) and $g' \neq 0$ on (a, b) , there is a $c \in (a, b)$ with

KEY

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

- Taking $g(x) = x$ recovers Lagrange's theorem.
- It is the statement behind L'Hôpital's rule.
- Advanced only; JEE Main lists Rolle and Lagrange alone.

Not two separate points. The same c serves both derivatives. Applying Lagrange to f and to g separately gives two different interior points and does not prove this.

93. WHAT THE MEAN VALUE THEOREMS BUY YOU B-13

- $f' = 0$ on an interval $\Rightarrow f$ is constant there.
- $f' = g'$ on an interval $\Rightarrow f - g$ is constant there.
- $m \leq f'(x) \leq M$ on $[a, b] \Rightarrow m(b - a) \leq f(b) - f(a) \leq M(b - a)$.
- $|f'(x)| \leq K \Rightarrow |f(x) - f(y)| \leq K|x - y|$ for all x, y in the interval.

For $0 < a < b$, applying LMVT to $\tan^{-1} x$ gives $\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$.

Bounding a difference is the giveaway. Any question that asks you to sandwich $f(b) - f(a)$, or to prove $|f(x) - f(y)|$ is small, is an LMVT question with the bounds on f' supplied in the statement.

94. COUNTING ROOTS WITH ROLLE B-14

- If f has n distinct real roots then f' has at least $n - 1$ real roots, one strictly between each consecutive pair.
- If f' has no real root, f has at most one; combined with a sign change (IVT) that makes exactly one.
- Existence:** f continuous with $f(a)(b) < 0$ gives at least one root in (a, b) . **Uniqueness:** show f' never vanishes there.

HELPER FUNCTIONS

TO STUDY	APPLY ROLLE TO
$f'(x) + kf(x) = 0$	$h(x) = e^{kx} f(x)$
$x f'(x) + n f(x) = 0$	$h(x) = x^n f(x)$
$f'(x) - f(x) \tan x = 0$	$h(x) = f(x) \cos x$

Darboux's theorem. A derivative has the intermediate value property even when it is not continuous, so f' cannot jump across a value without taking it. That is why sign changes of f' are legitimate evidence of a zero.

C MONOTONICITY & CONCAVITY 15-23

95. WHAT INCREASING MEANS C-15

On an interval I , f is **strictly increasing** if

KEY

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \quad \text{for all } x_1, x_2 \in I$$

Look for the sign change, never the vanishing. A zero of f' coming from an even power does not flip the sign and gives no inflection; a zero from an odd power does. Inflection points may also occur where f'' fails to exist.

- Replace $<$ by $\leq$ on the right for **non-decreasing** (also called increasing in the weak sense).
- Strictly decreasing and non-increasing reverse the second inequality.
- Monotonic** means one of the four throughout the interval.

Monotonicity belongs to an interval, not to a point. "Increasing at $x = 2$ " is not a statement the definition can test, because it compares two different inputs. Questions phrase it as "increasing on" for that reason.

96. THE DERIVATIVE TEST C-16

Let f be continuous on $[a, b]$ and differentiable on (a, b) .

- $f' > 0$ on $(a, b) \Rightarrow f$ strictly increasing on $[a, b]$.
- $f' < 0$ on $(a, b) \Rightarrow f$ strictly decreasing on $[a, b]$.
- $f' = 0$ on $(a, b) \Rightarrow f$ constant on $[a, b]$.

THE THREE STATEMENTS, SIDE BY SIDE

- $f' > 0$ on $I \Rightarrow$ strictly increasing. **True**, and this is the workhorse.
- strictly increasing $\Rightarrow f' \geq 0$. **True**.
- strictly increasing $\Rightarrow f' > 0$. **False**. $f(x) = x^3$ is strictly increasing and $f'(0) = 0$.

Zeros of f' are allowed if they are isolated. If $f' \geq 0$ on I and f' does not vanish on any whole subinterval, f is still strictly increasing. This is what makes x^3 legal and what parameter questions actually demand.

97. THE WORKING METHOD C-17

- Fix the domain of f first; monotonicity is only claimed inside it.
- Differentiate and factorise f' completely.
- Mark the zeros of f' and the points where f' fails to exist on a number line.
- Read the sign of f' in each interval from the factors, using the even-power rule.
- Report intervals, in the domain, with the correct kind of bracket.

The even-power rule. A factor to an even power never changes the sign of f' ; it only touches zero. Only odd-power factors flip it. The same rule reappears for f'' in section 20.

98. MONOTONIC FOR EVERY VALUE OF A PARAMETER C-18

"Find all a for which f is increasing on $\mathbb{R}$ " is a quadratic-sign question in disguise. Demand $f'(x) \geq 0$ for all x .

KEY

$$px^2 + qx + r \geq 0 \text{ for all } x \Leftrightarrow p > 0, q^2 - 4pr \leq 0 \text{ or } p = q = r, r \geq 0$$

$f(x) = x^3 + ax^2 + 3x + 7$; $f' = 3x^2 + 2ax + 3 \geq 0$ needs $4a^2 - 36 \leq 0$, so $a \in [-3, 3]$.

Why $\geq$ and not $>$. A non-zero quadratic cannot vanish on a whole interval, so its isolated zeros are harmless and section 16 still delivers strict increase. Demanding $f' > 0$ everywhere would wrongly exclude the end points $a = \pm 3$. If the parameter sits in the leading coefficient, test $p = 0$ separately; that degenerate case is usually an answer, not an exclusion.

99. CONCAVITY AND THE SECOND DERIVATIVE C-19

f is **concave up** on I when f' is increasing there, and **concave down** when f' is decreasing. Hence, for twice differentiable f :

KEY

$$f'' > 0 \Rightarrow \text{concave up} \\ f'' < 0 \Rightarrow \text{concave down}$$

- Concave up means the curve lies **above** each of its tangents; concave down, below them.
- The signs of f' and f'' are independent, so all four combinations occur.
- Jensen:** if $f'' > 0$ on $[a, b]$ then $f(\frac{x_1 + x_2}{2}) < \frac{f(x_1) + f(x_2)}{2}$ for $x_1 \neq x_2$.

Rising and bending are different questions. A function can rise while bending either way. Statement-based questions are built by naming one sign and asking about the other.

D POINTS OF INFLECTION 24-28

x_0 is a point of **inflection** if the concavity of f changes as x passes through x_0 .

KEY

- If f'' is continuous at an inflection point then $f''(x_0) = 0$.
- Conversely, f' **changing sign** at x_0 makes x_0 an inflection point.
- $f''(x_0) = 0$ alone is **not** enough: $f(x) = x^4$ has $f''(0) = 0$ and no inflection.

Look for the sign change, never the vanishing. A zero of f'' coming from an even power does not flip the sign and gives no inflection; a zero from an odd power does. Inflection points may also occur where f'' fails to exist.

96. SECOND AND n TH ORDER TESTS D-26

If $f'(c) = 0$ and $f''(c)$ exists: $f''(c) < 0$ gives a maximum, $f''(c) > 0$ a minimum, and $f''(c) = 0$ gives **no information**.

n TH ORDER TEST

If $f'(c) = f''(c) = \dots = f^{(n-1)}(c) = 0$ and $f^{(n)}(c) \neq 0$:

- even** and $f^{(n)}(c) < 0$: local maximum.
- even** and $f^{(n)}(c) > 0$: local minimum.
- odd** (so $n \geq 3$, since $f'(c) = 0$): neither; c is a point of inflection.

Why the parity decides. Near c , $f(x) - f(c) \approx \frac{f^{(n)}(c)}{n!} (x - c)^n$. An even power keeps its sign on both sides, so the value at c is extreme; an odd power changes sign, so it is not.

97. PROVING ONE-ONE, AND PROVING INEQUALITIES D-21

One-one. If f is strictly monotonic on an interval it is injective there, so a strict sign for f' **throughout an interval** settles injectivity in one line. The domain must be a single interval: $\tan x$ has $f' > 0$ at every point of its domain and is still not one-one, because the derivative says nothing about a break.

THE AUXILIARY FUNCTION METHOD

- To prove $g(x) > h(x)$ for $x > a$, set $F(x) = g(x) - h(x)$.
- Show $F(a) \geq 0$ at the left end.
- Show $F'(x) \geq 0$ on the interior, vanishing only at isolated points.
- Then F increases strictly from a non-negative start, so $F > 0$ for every $x > a$.

$\sin x < x$ for $x > 0$: take $F(x) = x - \sin x$, $F(0) = 0$, $F'(x) = 1 - \cos x \geq 0$ and zero only at isolated points.

Check the end value before differentiating. A positive derivative proves nothing on its own; the inequality is anchored by the value at the left end, and forgetting that anchor is the usual gap in these proofs.

98. HOW MANY ROOTS D-22

- Write the equation as $f(x) = 0$ and differentiate.
- Find the intervals of monotonicity from the sign of f' ; on each one f can cross zero at most once.
- Evaluate f at the ends of each interval, including the limits at $\pm\infty$.
- A sign change on a monotone stretch is exactly one root; with no sign change the only possible root is an end of the stretch, where f may itself be zero.

For an equation of the form $g(x) = h$, the count is the number of times the horizontal line $y = h$ meets the graph, which the turning values decide.

Separate the constant. Rewriting $g(x) - k = 0$ as $g(x) = k$ turns a root count into a picture: sketch g once, then read off the answer for every k the question asks about.

99. MONOTONICITY OF COMPOSITES AND INVERSES D-23

- increasing $\cdot$ increasing = increasing; decreasing $\cdot$ decreasing = increasing.
- increasing $\cdot$ decreasing = decreasing, and so is the other order.
- If f is strictly increasing then f^{-1} exists on the range and is strictly increasing; a strictly decreasing f has a strictly decreasing inverse.
- f increasing and g increasing $\Rightarrow f \circ g$ increasing; nothing general holds for fg unless both are positive.

Applying an increasing function to both sides of an inequality preserves it; applying a decreasing one reverses it:

KEY

$$a < b \Rightarrow \ln a < \ln b \quad (a, b > 0)$$

This is how logarithms get taken safely. A maximum of f is a maximum of $\ln f$ whenever $f > 0$, which turns a product into a sum and often removes the need for the product rule entirely.

E MAXIMA & MINIMA 24-33

96. WHAT AN EXTREMUM IS D-24

f has a **local maximum** at c if $f(c) \geq f(x)$ for all x in some neighbourhood of c ; a local minimum reverses the inequality.

- The definition compares values only. It says nothing about continuity or differentiability.
- Critical point:** a point of the domain where $f'(c) = 0$ or $f'(c)$ does not exist.
- Fermat:** if f has a local extremum at an interior c and $f'(c)$ exists, then $f'(c) = 0$.
- The converse fails: $f'(c) = 0$ does not make c an extremum (x^3 at 0).

Critical points are candidates, not answers. Every interior extremum is a critical point, so the list of critical points is where to look — but each one still has to be tested or compared.

97. THE FIRST DERIVATIVE TEST D-25

Let f be continuous at c , let c be a critical point, and examine the sign of f' on either side:

- $+$ $\rightarrow$ $-$: local maximum at c .
- $-$ $\rightarrow$ $+$: local minimum at c .
- no change of sign: c is **not** an extremum. At a stationary point of a smooth curve that is the horizontal inflection you expect; at a corner it is simply neither, so confirm an inflection from a sign change of f'' (section 20), never from this line alone.

This test asks least of f . It needs no second derivative and it works at corners and wherever f' fails to exist, which is exactly where the second derivative test is silent — provided f is continuous at c and f' keeps one sign on each side. Continuity at c is not optional: without it the signs describe the neighbours, not the value at c .

96. SECOND AND n TH ORDER TESTS D-26

If $f'(c) = 0$ and $f''(c)$ exists: $f''(c) < 0$ gives a maximum, $f''(c) > 0$ a minimum, and $f''(c) = 0$ gives **no information**.

n TH ORDER TEST

If $f'(c) = f''(c) = \dots = f^{(n-1)}(c) = 0$ and $f^{(n)}(c) \neq 0$:

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- odd** (so $n \geq 3$, since $f'(c) = 0$): neither; c is a point of inflection.

Why the parity decides. Near c , $f(x) - f(c) \approx \frac{f^{(n)}(c)}{n!} (x - c)^n$. An even power keeps its sign on both sides, so the value at c is extreme; an odd power changes sign, so it is not.

97. EXTREMA WHERE f' IS NOT DIFFERENTIABLE D-27

Nothing in the definition mentions a derivative, so corners, jumps and isolated points can be extrema.

Look at the value, not the derivative. A corner, a jump or an isolated point can be an extremum, and section 09 lists where such points come from. Reaching for f' first at such a point answers a question the definition never asked.

98. GLOBAL MAXIMUM AND MINIMUM D-28

On a closed interval $[a, b]$ a continuous f attains both a greatest and a least value, and each occurs at a critical point inside or at an end point.

KEY

- Confirm f is continuous on $[a, b]$.
- List a , b and every critical point inside (a, b) .
- Evaluate f at all of them.

The largest value found is the global maximum; the smallest is the global minimum.

Evaluate, do not classify. No second derivative test is needed here. Classifying each critical point wastes time, because the final comparison of values settles everything by itself.

99. FUNCTIONS DEFINED BY AN INTEGRAL D-29

For f continuous and g differentiable, the Newton-Leibnitz formula gives

KEY

$$\frac{d}{dx} \int_a^x h(t) dt = f(x) = f(h(x))h'(x) - f(g(x))g'(x)$$

- You never evaluate the integral; you differentiate it and read off the sign.
- The zeros of that derivative are the critical points of the integral function.
- The same formula converts an integral equation into a differential equation.

$F(x) = \int_x^3 (t-1)(t-3) dt$ has $F'(x) = (x-1)(x-3)$, a maximum at $x = 1$ and a minimum at $x = 3$.

Do not forget the lower limit. When both limits vary, the lower one contributes with a minus sign and its own chain-rule factor; dropping it is the standard slip in this family.

96. ROOTS OF A CUBIC FROM ITS TURNING VALUES D-30

For $f(x) = x^3 + ax^2 + bx + c$, let $D = 4a^2 - 12b$ be the discriminant of $f'(x) = 3x^2 + 2ax + b$.

- $D < 0$: $f' > 0$ everywhere, f strictly increasing, exactly one real root and no extremum.
- Critical point:** a point of the domain where $f'(c) = 0$ or $f'(c)$ does not exist.
- Fermat:** if f has a local extremum at an interior c and $f'(c)$ exists, then $f'(c) = 0$.
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