

CONCEPT MAP

DIFFERENTIATION

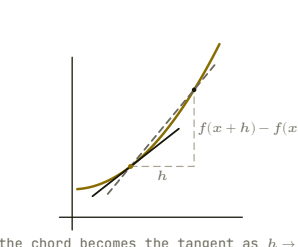
5 TOPICS • 34 CORE IDEAS • ONE SHEET



A THE RULES & THE TABLE 01-07

01 THE DERIVATIVE FROM FIRST PRINCIPLES A-01

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$



- The quotient is the slope of the chord; the limit is the slope of the tangent.
- Equivalently $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$.
- The derivative exists only when that limit exists and is finite.

First principles is examined, not just quoted. Questions that hand you a functional equation, or that define f only through a limit, are asking for this definition rather than for the table in section 02.

02 STANDARD DERIVATES: ALGEBRAIC AND EXPONENTIAL A-02

f	f'
x^n	nx^{n-1}
e^x	e^x
a^x	$a^x \ln a, a > 0$
$\ln x$	$\frac{1}{x}, x > 0$
$\ln x $	$\frac{1}{x}, x \neq 0$
$\log_a x$	$\frac{1}{x \ln a}, x > 0$
$ x $	$\frac{x}{ x }, x \neq 0$

The power rule holds for every real n on $x > 0$. For integer n it holds for all $x \neq 0$, and for a whole number $n \geq 1$ for every x . At $x = 0$ a fractional power such as $x^{2/3}$ is defined but has no derivative, so being in the domain is not enough. Rewrite roots and reciprocals as powers before differentiating and the rule does the rest.

03 STANDARD DERIVATES: TRIGONOMETRIC A-03

f	f'
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

Every co-function carries a minus sign. Cosine, cotangent and cosecant all differentiate with a minus, and the other three without one. That single observation halves what has to be remembered here.

04 STANDARD DERIVATES: INVERSE TRIGONOMETRIC A-04

f	f'
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}, x < 1$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$-\frac{1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{ x \sqrt{x^2-1}}, x > 1$
$\operatorname{cosec}^{-1} x$	$-\frac{1}{ x \sqrt{x^2-1}}$

The modulus in the last two is not optional. $\sec^{-1}$ is increasing on both branches, so its derivative is positive on both, which only the $|x|$ delivers. Dropping it gives the wrong sign for every $x < -1$.

05 PRODUCT AND QUOTIENT A-05

KEY

$$(uv)' = u'v + uv'$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}, \quad v \neq 0$$

- For three factors, $(uvw)' = u'vw + uv'w + uvw'$: differentiate one at a time and add.
- The quotient rule is the product and chain rules applied to $u \cdot v^{-1}$.
- The order in the numerator matters: it is $u'v - uv'$, never the other way round.

Simplify before differentiating. A quotient that cancels, or a product that expands to a polynomial, is faster done by hand than by rule. The rules are for when the algebra will not simplify, not a first resort.

06 THE CHAIN RULE A-06

KEY

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

- For $y = f(g(x))$: $\frac{dy}{dx} = f'(g(x))g'(x)$.
- For more links, multiply one factor for each: $\frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$.
- Work from the outside in, and stop only when the innermost variable is x itself.

$$y = \sin^3(2x+1); \quad \frac{dy}{dx} = 3\sin^2(2x+1) \cdot \cos(2x+1) \cdot 2.$$

Count the links before you start. The commonest error is stopping one factor early, which is invisible in the answer because the result still looks like a derivative. Write the chain out and tick off each link.

07 DERIVATIVE OF ONE FUNCTION WITH RESPECT TO ANOTHER A-07

To differentiate $f(x)$ with respect to $g(x)$ rather than with respect to x :

KEY

$$\frac{df}{dg} = \frac{df/dx}{dg/dx}, \quad \frac{dg}{dx} \neq 0$$

The derivative of $\sin^{-1} x$ with respect to $\cos^{-1} x$ is $\frac{1/\sqrt{1-x^2}}{-1/\sqrt{1-x^2}} = -1$.

Both derivatives are taken with respect to x first. The phrase "differentiate f with respect to g " never means substituting one into the other. Differentiate each separately and divide, and simplify both before dividing.

B THE SPECIAL TECHNIQUES 08-17

08 IMPLICIT DIFFERENTIATION B-08

- Differentiate every term with respect to x .
- Each y term picks up a factor $\frac{dy}{dx}$ by the chain rule.
- Collect the $\frac{dy}{dx}$ terms on one side and solve.

$$x^2 + xy + y^2 = 7 \text{ gives } 2x + y + xy' + 2yy' = 0, \text{ so } y' = -\frac{2x+y}{x+2y}.$$

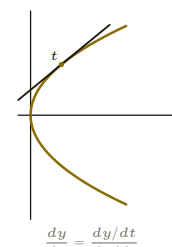
A product of x and y needs the product rule as well. The term xy differentiates to $y + xy'$, not to xy' . That single omission is the commonest mistake in this technique.

09 PARAMETRIC DIFFERENTIATION B-09

If $x = f(t)$ and $y = g(t)$, then

KEY

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{dx}{dt} \neq 0$$



$$x = at^2, y = 2at \text{ gives } \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}, \text{ the standard parabola revisited.}$$

The answer is a function of t , and that is allowed. Convert to x only if the question asks for it. Where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$ the curve has a vertical tangent, and the formula reports that as an infinite slope. Where both vanish the point is singular and the tangent has to be found by a limit.

10 THE SECOND DERIVATIVE OF A PARAMETRIC CURVE B-10

KEY

$$\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{1}{\frac{dx}{dt}}$$

Differentiate the first derivative with respect to t , then divide by $\frac{dx}{dt}$ again.

It is not $\frac{d^2y/dt^2}{d^2x/dt^2}$. That expression is wrong and is printed among the options in almost every question on this topic. The second division by $\frac{dx}{dt}$ is the whole content of the correct formula.

11 LOGARITHMIC DIFFERENTIATION B-11

Take logarithms of both sides, then differentiate implicitly. Use it for a power with a variable in the exponent, and for long products and quotients.

KEY

$$\frac{d}{dx} [f^g] = f^g \left[g' \ln f + \frac{g f'}{f} \right], \quad f > 0$$

- $\frac{d}{dx} x^x = x^x(1 + \ln x)$ on $x > 0$.
- For a product of many factors, $\frac{y'}{y}$ is the sum of the individual $\frac{f'_i}{f_i}$ terms.
- For a quotient, those terms are subtracted.

The result splits into two recognisable pieces. One treats the exponent as constant and one treats the base as constant. If a solution shows only one of them, it has differentiated f^v as if y or f were a number.

12 POWERS, AND WHICH RULE APPLIES B-12

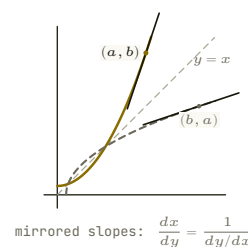
FORM	RULE
x^n, n constant	power rule: nx^{n-1}
a^x, a constant	exponential rule: $a^x \ln a$
$[f(x)]^n$	chain rule on the power
$a^{f(x)}$	chain rule on the exponent
$[f(x)]^{g(x)}$	logarithms, section 11

Check what is variable before choosing. x^x is neither a power nor an exponential, and treating it as either gives a wrong answer that looks reasonable. Only the fifth row applies when the variable sits in both places.

13 DERIVATIVE OF AN INVERSE FUNCTION B-13

KEY

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}, \quad \frac{dy}{dx} \neq 0$$



$$\text{If } g = f^{-1} \text{ and } b = f(a), \text{ then } g'(b) = \frac{1}{f'(a)}.$$

The reciprocal is of the derivative, not of the function. Evaluate f' at the point a of the original curve, not at b . A question that gives f and asks for $(f^{-1})'$ at a value is asking you to find which a maps there first.

14 SIMPLIFY INVERSE TRIGONOMETRIC FORMS FIRST B-14

EXPRESSION	FOLDS TO
$\tan^{-1} \frac{2x}{1-x^2}$	$2 \tan^{-1} x$ on $ x < 1$
$\sin^{-1} \frac{2x}{1+x^2}$	$2 \tan^{-1} x$ on $ x \leq 1$
$\cos^{-1} \frac{1-x^2}{1+x^2}$	$2 \tan^{-1} x$ on $x \geq 0$
$\tan^{-1} \sqrt{\frac{1-\cos x}{1+\cos x}}$	$\frac{x}{2}$ on $0 < x < \pi$
$\sin^{-1} (3x - 4x^3)$	$3 \sin^{-1} x$ on $ x \leq \frac{1}{3}$

The interval is part of the identity. The derivative of the raw form is right on one interval and wrong on the next, because the folded form changes there. Fold first, note where the folding is valid, and only then differentiate.

15 SUBSTITUTIONS THAT FOLD AN EXPRESSION B-15

SEEING	PUT
$\sqrt{1-x^2}$	$x = \sin \theta$
$\sqrt{1+x^2}$ or $1+x^2$	$x = \tan \theta$
$\sqrt{x^2-1}$	$x = \sec \theta$
$\sqrt{\frac{1-x}{1+x}}$	$x = \cos 2\theta$
$\sqrt{a^2-x^2}$	$x = a \sin \theta$

Substituting turns the expression into a single angle. The inverse function then cancels the direct one, the whole expression collapses to a multiple of θ , and differentiating what is left takes one line instead of half a page.

16 DIFFERENTIATING A DETERMINANT B-16

Differentiate one row at a time and add the results. For a 3×3 determinant that is three determinants, not one.

KEY

$$\frac{d}{dx} \begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix} = \begin{vmatrix} f'_1 & f_2 \\ g_1 & g_2 \end{vmatrix} + \begin{vmatrix} f_1 & f'_2 \\ g_1 & g'_2 \end{vmatrix}$$

- The same rule works column by column; use whichever has fewer variable entries.
- A row of constants differentiates to a row of zeros, so that term vanishes.
- A determinant is linear in each row separately, which is why the product rule applies one row slot at a time.

Never expand the determinant first. Expanding a 3×3 gives six products to differentiate; the row rule gives three determinants, of which some usually vanish at once.

17 FUNCTIONS DEFINED BY AN EQUATION THEY SATISFY B-17

When f is given only through a functional equation, differentiate it or use first principles.

- Substitute convenient values to find $f(0)$ or $f(1)$.
- Differentiate both sides with respect to one variable, holding the other fixed.
- Set the held variable to a convenient value to obtain a differential equation in f .

$f(x+y) = f(x)f(y)$ with $f \neq 0$ forces $f(0) = 1$; then $f'(0) = k$ gives $f'(x) = kf(x)$, so $f(x) = e^{kx}$. The zero function is the only other solution.

Which variable you differentiate in decides how far you get. Differentiating with respect to y and then putting $y = 0$ is the move that usually produces a first order equation, because $f(0)$ and $f'(0)$ are the two things the question has given you.

C HIGHER ORDER 18-23

18 SUCCESSIVE DIFFERENTIATION C-18

Write $y_n, f^{(n)}(x)$ or $\frac{d^ny}{dx^n}$ for the n th derivative.

- Each derivative is taken of the previous one, so simplify at every stage.
- A polynomial of degree n with leading coefficient a_n has $y_n = n!a_n$ and $y_{n+1} = 0$.
- For an implicit relation, differentiate the whole equation again rather than the solved form.

Look for a pattern after two or three steps. Most n th derivative questions are answered by spotting the pattern and proving it by induction, not by differentiating n times.

19 THE n TH DERIVATIVES WORTH KNOWING C-19

- $\frac{d^n}{dx^n} e^{ax} = a^n e^{ax}$
- $\frac{d^n}{dx^n} \sin x = \sin\left(x + \frac{n\pi}{2}\right)$
- $\frac{d^n}{dx^n} \cos x = \cos\left(x + \frac{n\pi}{2}\right)$
- $\frac{d^n}{dx^n} \ln x = \frac{(-1)^{n-1}(n-1)!}{x^n}$
- $\frac{d^n}{dx^n} \frac{1}{(ax+b)^{n+1}} = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$
- $\frac{d^n}{dx^n} x^m = \frac{m!}{(m-n)!} x^{m-n}$ for a positive integer m and $n \leq m$

Adding $\pi/2$ is what differentiating a sine does. Each derivative advances the phase by a quarter turn, which is why the pattern has period four. The same reading gives the cosine line without memorising it separately.

20 THE GENERAL LEIBNIZ FORMULA C-20

KEY

$$(uv)^{(n)} = \sum_{r=0}^n {}^nC_r u^{(n-r)} v^{(r)}$$

- It is the binomial theorem with derivatives in place of powers.
- Choose v to be the factor that dies after a few derivatives, usually the polynomial.
- If v is a polynomial of degree m , the sum has at most $m+1$ non-zero terms.

$(x^2 e^x)^{(n)} = e^x (x^2 + 2nx + n(n-1))$, since the x^2 factor contributes only three terms.

Advanced only, and the choice of v is the whole method. Taking the exponential as v makes the sum run to n terms and go nowhere. Taking the polynomial as v ends it in three.

Use the implication in its contrapositive form. Not continuous means not differentiable, which is the direction the questions use. Arguing the other way round proves nothing.

21 RECURRENCE RELATIONS FOR y_n C-21

- Differentiate the given relation once or twice to remove any root or fraction.
- Apply the Leibniz formula to the resulting product, n times.
- The result is a recurrence in y_{n+2}, y_{n+1} and y_n .
- Put $x = 0$ to get a numerical recurrence, then read off the values.

$y = (\sin^{-1} x)^2$ satisfies $(1-x^2)y_2 - xy_1 = 2$, and differentiating that n times by Leibniz gives $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$ for $n \geq 1$.

Clear the root before applying Leibniz. The formula needs a product of two clean factors. One preliminary differentiation, followed by multiplying through by the denominator, is almost always what produces that shape.

22 DIFFERENTIATING AN INTEGRAL C-22

For f continuous and g, h differentiable,

KEY

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) dt = f(h(x))h'(x) - f(g(x))g'(x)$$

- Constant limits give derivative zero, because the integral is a number.
- The lower limit carries a minus sign and its own chain factor.
- This is how a function defined by an integral is differentiated without evaluating it.

The rule converts an integral condition into a differential equation. That conversion is the standard opening move for the paragraph questions in Advanced, and the sheets on definite integration and differential equations both use it.

23 HIGHER DERIVATIVES OF IMPLICIT AND PARAMETRIC FORMS C-23

- Implicit: differentiate the whole equation a second time, substitute the known y' , then solve for y'' .
- Parametric: use the formula of section 10, dividing by $\frac{dx}{dt}$ a second time.
- Inverse: $\frac{d^2x}{dy^2} = -\frac{d^2y/dx^2}{(dy/dx)^3}$

The inverse second derivative is not a reciprocal either. Only the first derivative inverts by taking a reciprocal. The second picks up a cube in the denominator and a minus sign, both of which the printed options will test.

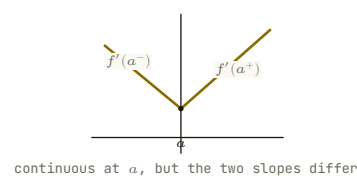
D DIFFERENTIABILITY 24-30

24 DIFFERENTIABILITY AT A POINT D-24

f is differentiable at a when both one-sided derivatives exist, are finite and are equal.

KEY

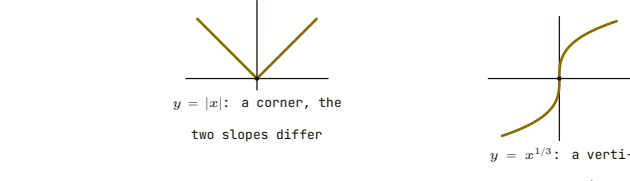
$$f'(a^-) = \lim_{h \rightarrow 0^+} \frac{f(a-h) - f(a)}{-h}$$
$$f'(a^+) = \lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h}$$



Check continuity first. If f is not continuous at a it cannot be differentiable there, and the one-sided derivatives need not be computed at all. That check takes one line and settles many questions on its own.

25 DIFFERENTIABLE IMPLIES CONTINUOUS D-25

If f is differentiable at a then f is continuous at a . The converse is false.

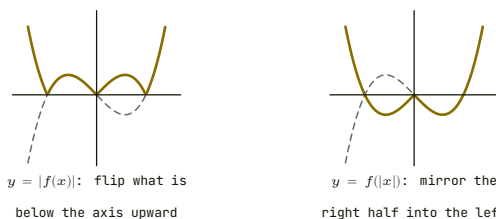


- A corner is continuous with unequal one-sided derivatives.
- A vertical tangent has both one-sided derivatives $+\infty$, as $x^{1/3}$ does; a cusp has $+\infty$ on one side and $-\infty$ on the other, as $|x|^{1/3}$ does.
- Continuity is about the value, differentiability about the slope, so the second is the stronger condition.

Use the implication in its contrapositive form. Not continuous means not differentiable, which is the direction the questions use. Arguing the other way round proves nothing.

26 THE MODULUS AND THE SHARP CORNER RULE D-26

- If $f(a) = 0$ and $f'(a) \neq 0$, then $|f|$ is not differentiable at a .
- If $f(a) = 0$ and $f'(a) = 0$, then $|f|$ is differentiable at a .
- Away from the zeros of f , $|f|$ inherits the differentiability of f .



List the roots, then test each one. "How many points of non-differentiability does $|f(x)|$ have" is answered by solving $f = 0$ and checking f' at each root, not by drawing the graph and counting the kinks by eye.

27 MAXIMUM AND MINIMUM OF TWO FUNCTIONS D-27

For differentiable f and g , $\max\{f, g\}$ and $\min\{f, g\}$ can fail to be differentiable only where $f = g$.

KEY

$$\max\{f, g\} = \frac{f+g+|f-g|}{2}, \quad \min\{f, g\} = \frac{f+g-|f-g|}{2}$$

- The identity turns the question into the modulus rule of section 26 applied to $f-g$.
- At a crossing where $f' \neq g'$ there is a corner, so the function is not differentiable.
- At a crossing where $f' = g'$ the two graphs touch smoothly and it is differentiable.

Solve $f = g$ first. Those points are the only candidates, and everywhere else the maximum simply equals whichever function is larger, inheriting its derivative.

28 PIECEWISE RULES AT A JUNCTION D-28

- Check the two one-sided limits and the value: continuity first.
- If continuous, compute $f'(a^-)$ and $f'(a^+)$ from the two formulae.
- For an ordinary formula the power rule may be used on each side; for a modulus or a greatest integer, go back to the definition.
- Equal and finite means differentiable; anything else does not.

Differentiating each piece is not enough. The derivatives of the two formulae may agree at a while the function is still not differentiable there, because continuity failed. The order of the two checks is what makes the argument valid.

29 DARBOUX'S THEOREM D-29

A derivative has the intermediate value property even when it is not continuous: if f is differentiable on $[a, b]$ then f' takes every value between $f'(a)$ and $f'(b)$.

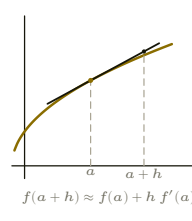
- So a derivative cannot have a jump discontinuity.
- A function with a jump in it is therefore not the derivative of anything.
- This is what makes a sign change of f' legitimate evidence of a zero of f' .

Advanced only, and usually asked in reverse. The question shows a piecewise constant function and asks whether it can be some f' . By this theorem it cannot, and no calculation is needed.

30 DIFFERENTIALS AND APPROXIMATION D-30

KEY

$$\Delta y \approx f'(x) \Delta x, \quad f(a+h) \approx f(a) + h f'(a)$$



- Choose a to be the nearest point where f is easy to evaluate exactly.
- The percentage error in $y = x^n$ is about $|n|$ times that in x , with the opposite sign when $n < 0$.
- Taking logarithms first turns any power-product relation into a sum of relative errors.

The tangent line is the approximation. That is why it is accurate only for small h , and why it fails near a sharp bend where the curve leaves its tangent quickly.