

CONCEPT MAP

INDEFINITE INTEGRATION

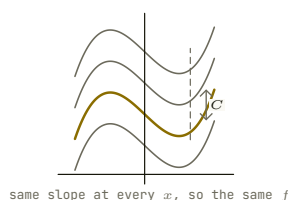
5 TOPICS · 35 CORE IDEAS · ONE SHEET



A STANDARD FORMS 01-07

01 THE ANTIDERIVATIVE AND THE CONSTANT
 F is an antiderivative of f when $F' = f$. On any **interval**, every antiderivative differs from F by a constant, so

$$\int f(x) dx = F(x) + C$$



- The answer is a **family** of curves, each a vertical translate of the others.
- When the domain falls into separate intervals, as for $1/x$, there is one such constant on each piece.
- $\frac{d}{dx} \int f(x) dx = f(x)$, and $\int f'(x) dx = f(x) + C$.
- Integration is linear: $\int (\lambda f + \mu g) = \lambda \int f + \mu \int g$.

Two answers can differ and both be right. $\int 2 \sin x \cos x dx$ is $\sin^2 x + C$, $-\cos^2 x + C$ and $-\frac{1}{2} \cos 2x + C$ at once. They differ by a constant, so an option that does not match yours may still be the same antiderivative. Differentiate the option to settle it.

02 STANDARD INTEGRALS: ALGEBRAIC AND EXPONENTIAL A-02

f	$\int f dx$
x^n	$\frac{x^{n+1}}{n+1}, n \neq -1$
$\frac{1}{x}$	$\ln x $
e^x	e^x
a^x	$\frac{a^x}{\ln a}, a > 0, a \neq 1$
$\ln x$	$x \ln x - x, x > 0$

The modulus in $\ln|x|$ is not decoration. $1/x$ is defined on both sides of the origin and the antiderivative must be too. Dropping the bars makes the answer wrong for every negative x , and papers do set the negative branch.

03 STANDARD INTEGRALS: TRIGONOMETRIC A-03

f	$\int f dx$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$\ln \sec x $
$\cot x$	$\ln \sin x $
$\sec x$	$\ln \sec x + \tan x $
$\csc x$	$\ln \csc x - \cot x $
$\sec^2 x$	$\tan x$
$\csc^2 x$	$-\cot x$
$\sec x \tan x$	$\sec x$
$\csc x \cot x$	$-\csc x$

Every row is a derivative read backwards. If a row will not come to mind, differentiate the right column instead of trying to recall the left. The four that need memorising are tan, cot, sec and cosec; the rest fall out of the derivative table.

04 THE SIX FORMS A-04

- $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$
- $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right|$
- $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right|$
- $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$
- $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln \left| x + \sqrt{x^2 \pm a^2} \right|$
- $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{x}{a} \right|$

Read the root first, then the sign. With no root, a plus gives an inverse tangent and a minus gives a logarithm of a quotient. Under a root, a minus on x^2 gives an inverse sine, while a plus, or a minus on a^2 , gives $\ln|x \pm \sqrt{\quad}|$. Row 6 is the odd one out and is simply memorised. Rows 2 and 3 are negatives of each other, so getting the order of a and x wrong flips the sign of the whole answer.

05 THE THREE SURDS A-05

KEY

$$\int \sqrt{a^2 - x^2} dx = \frac{\pi}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$$
$$\int \sqrt{x^2 + a^2} dx = \frac{\pi}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln|x + \sqrt{x^2 + a^2}|$$
$$\int \sqrt{x^2 - a^2} dx = \frac{\pi}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln|x + \sqrt{x^2 - a^2}|$$

Each is one integration by parts away from a form in section 04. Take $u = \sqrt{\quad}$ and $v = 1$. The leftover splits into the original integral plus the corresponding row of the six forms, so collect the two copies and divide by two, exactly as in section 18. That is also the fastest way to repair a half-remembered sign in the second term.

06 LINEAR SUBSTITUTION A-06

If $\int f(x) dx = F(x) + C$, then for constants $a \neq 0$ and b ,

KEY

$$\int f(ax + b) dx = \frac{1}{a} F(ax + b) + C$$

- This turns every row of sections 02 to 05 into a family of results at no cost.
- $\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)}$ for $n \neq -1$, and $\frac{1}{a} \ln|ax + b|$ for $n = -1$.
- The rule works only when the inside is **linear**. There is no such formula for $f(x^2)$.

The $1/a$ is the whole content of the rule. It is the commonest missing factor in the chapter, and it is missing exactly when $a \neq 1$, which is when a paper bothers to set the question.

07 COMPLETING THE SQUARE A-07

- For $\int \frac{dx}{ax^2 + bx + c}$ and $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$:
- Take out the coefficient a from the quadratic.
 - Complete the square: $x^2 + \frac{b}{a}x + \frac{c}{a} = \left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}$.
 - Substitute $t = x + \frac{b}{2a}$, which is a linear substitution and costs nothing.
 - The integral is now one of the six forms of section 04.

The sign of the constant chooses the form, but only without a root. For $\int \frac{dx}{ax^2 + bx + c}$, a positive $4ac - b^2$ gives an inverse tangent and a negative one gives a logarithm. Under a root it is the sign of a that decides: $a > 0$ always gives $\ln|t + \sqrt{t^2 + k}|$ and $a < 0$ always gives an inverse sine. Read the leading coefficient first, then the constant.

08 SUBSTITUTION 08-14

08 THE SUBSTITUTION RULE B-08

KEY

$$\int f(g(x))g'(x) dx = \int f(t) dt, \quad t = g(x)$$

- Choose t so that dt absorbs a factor already present in the integrand.
- Replace **every** x , including the one hiding in dx .
- Integrate in t , then substitute back and add C .

Nothing in x may survive. A leftover x means the substitution was the wrong one, not that it needs forcing. Either express that x through t or start again with a different choice.

09 THE PATTERNS WORTH RECOGNISING B-09

- $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$
- $\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + C, n \neq -1$
- $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + C$
- $\int e^{f(x)} f'(x) dx = e^{f(x)} + C$

$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| = \ln|\sec x| + C$, which is row three of section 03 derived rather than recalled.

Look for the derivative before choosing a method. If some factor of the integrand is the derivative of another part of it, the question is a one line substitution and every heavier technique is wasted effort.

10 TRIGONOMETRIC SUBSTITUTIONS B-10

EXPRESSION	PUT
$a^2 - x^2$	$x = a \sin \theta$ or $a \cos \theta$
$a^2 + x^2$	$x = a \tan \theta$ or $a \cot \theta$
$x^2 - a^2$	$x = a \sec \theta$ or $a \csc \theta$
$\sqrt{\frac{a-x}{a+x}}$ or its reciprocal	$x = a \cos 2\theta$
$\sqrt{(x-a)(b-x)}$	$x = a \cos^2 \theta + b \sin^2 \theta$
$\sqrt{(x-a)(x-b)}$	$x = a \sec^2 \theta - b \tan^2 \theta$
$\sqrt{\frac{x}{a-x}}$	$x = a \tan^2 \theta$
$\sqrt{\frac{a-x}{a-x}}$	$x = a \sin^2 \theta$

The surd names the substitution. Each row is chosen so that a Pythagorean identity clears the root in one step. Reading the shape of the expression is faster than trying substitutions, and the last three rows are the ones candidates forget.

11 SPLITTING A LINEAR NUMERATOR B-11

For $\int \frac{px + q}{ax^2 + bx + c} dx$ and $\int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx$, write

KEY

$$px + q = \lambda \frac{d}{dx}(ax^2 + bx + c) + \mu$$

- Compare coefficients to find $\lambda = \frac{p}{2a}$ and then μ .
- The λ piece is the f'/f pattern of section 09, giving a logarithm or a root.
- The μ piece is a constant over the quadratic, so complete the square as in section 07.

One question, two standard integrals. This split is what turns the general linear numerator into two things already on the sheet, which is why the syllabus lists it as a standard form in its own right.

12 RATIONALISING SUBSTITUTIONS B-12

INTEGRAND	PUT
$\frac{1}{(ax + b)\sqrt{cx + d}}$	$cx + d = t^2$
$\frac{1}{(ax^2 + b)\sqrt{cx + d}}$	$cx + d = t^2$
$\frac{1}{(ax + b)\sqrt{cx^2 + d}}$	$ax + b = \frac{1}{t}$
$\frac{1}{x^k \sqrt{ax^2 + b}}$	$x = \frac{1}{t}$
$\sqrt{ax + b}$ in the integrand	$ax + b = t^n$

The aim is always the same. Kill the root by making the thing under it a perfect power, or push the awkward factor into the numerator with a reciprocal substitution. Both moves leave a rational integrand, which section 22 then handles.

13 THE $x \pm 1/x$ FAMILY B-13

For integrands of the form $\frac{x^2 \pm 1}{x^4 + kx^2 + 1}$, divide the numerator and denominator by x^2 ;

KEY

$$\frac{1 \pm \frac{1}{x^2}}{x^2 + k + \frac{1}{x^2}}$$

- For the $+$ numerator put $t = x - \frac{1}{x}$, so $dt = \left(1 + \frac{1}{x^2}\right) dx$ and $x^2 + \frac{1}{x^2} = t^2 + 2$.
- For the $-$ numerator put $t = x + \frac{1}{x}$, so $dt = \left(1 - \frac{1}{x^2}\right) dx$ and $x^2 + \frac{1}{x^2} = t^2 - 2$.
- Either way the integral becomes one of the six forms.

Match the substitution to the numerator, not the denominator. The sign in $t = x \pm \frac{1}{x}$ is fixed by which of $1 \pm \frac{1}{x^2}$ sits on top. A quartic denominator with neither numerator is split as the sum and the difference first.

14 SUBSTITUTIONS WORTH KNOWING B-14

SEEING	TRY
$f(\ln x) \cdot \frac{1}{x}$	$t = \ln x$
$f(\sin x) \cos x$	$t = \sin x$
$f(\tan x) \sec^2 x$	$t = \tan x$
$f(e^x) e^x$	$t = e^x$
$f\left(x + \frac{1}{x}\right)$ with $1 - \frac{1}{x^2}$	$t = x + \frac{1}{x}$
$\frac{1}{\sqrt{x}} f(\sqrt{x})$	$t = \sqrt{x}$

Every row is the same observation. A composite function appears together with the derivative of its inside, so the substitution is written on the page already. Scanning for that pairing is the first thing to do with any unfamiliar integrand.

15 BY PARTS & REDUCTION 15-21

15 INTEGRATION BY PARTS C-15

KEY

$$\int u v dx = u \int v dx - \int (u' \int v dx) dx$$

$\int$ counts from highest $\int$

$\int$ inverse trigonometric
 $\int$ logarithmic
 $\int$ algebraic
 $\int$ trigonometric
 $\int$ exponential

Choose u by ILATE: whichever kind appears earlier in the list is u , and the other is integrated.

The choice is about what differentiating does. An inverse trigonometric or logarithmic factor becomes algebraic when differentiated, which is progress; an exponential never simplifies either way, so it is left to be integrated.

16 INTEGRATING A SINGLE FUNCTION C-16

When only one function is printed, take the second factor to be 1.

- $\int \ln x dx = x \ln x - x + C$
- $\int \sin^{-1} x dx = x \sin^{-1} x + \sqrt{1 - x^2} + C$
- $\int \tan^{-1} x dx = x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + C$
- $\int \sec^{-1} x dx = x \sec^{-1} x - \text{sgn}(x) \ln|x + \sqrt{x^2 - 1}| + C$, the sign coming from the $|x|$ in the derivative of $\sec^{-1}$

The invisible factor of 1 is the trick. Every inverse and logarithmic function on the syllabus is integrated this way. For the inverse functions the leftover is the f'/f or the $f'/\sqrt{f}$ pattern of section 09; for $\ln x$ it is simply $\int 1 dx$.

17 THE TWO PATTERNS TO SPOT C-17

KEY

$$\int e^{ax} [f(x) + f'(x)] dx = e^{ax} f(x) + C$$

$$\int [f(x) + x f'(x)] dx = x f(x) + C$$

- The first generalises to $\int e^{ax} [af(x) + f'(x)] dx = e^{ax} f(x) + C$.
- Both are the product rule read backwards, so verify by differentiating the right side.
- $\int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx = \frac{e^x}{x} + C$ is the standard instance.

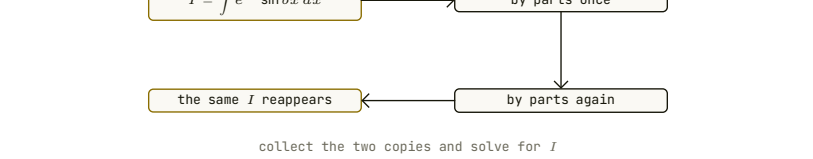
Look for a function beside its own derivative. If an e^x multiplies a bracket, test whether one term of the bracket is the derivative of the other. When it is, the answer is one line and no integration is done at all.

18 WHEN THE INTEGRAL RETURNS C-18

KEY

$$\int e^{ax} \sin bx dx = \frac{e^{ax} (a \sin bx - b \cos bx)}{a^2 + b^2} + C$$
$$\int e^{ax} \cos bx dx = \frac{e^{ax} (a \cos bx + b \sin bx)}{a^2 + b^2} + C$$

$$\frac{x^2 - 1}{x^2 - 1} = 1 + \frac{1}{x^2 - 1}, \text{ and the second piece is row 3 of section 04, giving } x + \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C.$$



Two rounds of parts, then algebra. The same integral reappears with a coefficient, so collect the copies and divide. Choosing a different factor for u on the second round undoes the first and returns the useless identity $I = I$.

19 REPEATED BY PARTS C-19

For $\int x^n e^{ax} dx$, $\int x^n \sin x dx$ and their relatives, differentiate the polynomial repeatedly;

KEY

$$\int x^n e^{ax} dx = e^{ax} (x^n - n x^{n-1} + n(n-1) x^{n-2} - \dots) + C$$

- The signs alternate and the chain terminates after n steps, when the polynomial is exhausted.
- Set the work out in two columns, derivatives of u against integrals of v , and read off the alternating products.
- For $\int x^n \sin x dx$ the second column cycles through $-\cos$, $-\sin$, $\cos$, $\sin$.

Differentiate the polynomial, never the other factor. Taking u to be the trigonometric or exponential factor makes the integral grow instead of shrink, and the chain then never terminates.

20 REDUCTION FORMULAE C-20

KEY

$$\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n-1} + \frac{n-1}{n} \int \sin^{n-2} x dx, n \neq 0$$
$$\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx, n \neq 1$$
$$\int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx, n \neq 1$$

$\int$ counts from highest $\int$

A recurrence is an answer. These questions usually ask for the relation between I_n and I_{n-2} , or for one specific n . Iterate down to I_0 or I_1 , which are x and $-\cos x$ for the sine family and x and $\ln|\sec x|$ for the tangent family.

21 BUILDING YOUR OWN REDUCTION C-21

- Split one factor off the power: write $\sin^n x$ as $\sin^{n-1} x \cdot \sin x$.
- Integrate by parts with the single factor as v .
- Use a Pythagorean identity to turn the leftover $\sin^{n-2} x \cos^2 x$ into $\sin^{n-2} x - \sin^n x$.
- The original integral has reappeared, so collect and divide.

This is the derivation, and it is worth knowing. The three formulae in section 20 are the cases the papers set, but a question on $\int x^n \ln x dx$ or $\int \cot^n x dx$ expects you to run these four steps yourself.

22 RATIONAL & TRIGONOMETRIC 22-32

FACTOR IN THE DENOMINATOR	CONTRIBUTES
$(x - a)$	$\frac{x - a}{x - a} = \frac{B}{(x - a)^2}$
$(x - a)^2$	$\frac{A}{x - a} + \frac{B}{(x - a)^2}$
$(x - a)^k$	one term for each power up to k
$x^2 + px + q$ irreducible	$\frac{Ax + B}{x^2 + px + q}$
$(x^2 + px + q)^2$	a linear numerator over each power

Find what you can by substitution, then compare coefficients for the rest. Putting $x = a$ into the cleared identity kills every term but the one over the highest power of $(x - a)$. The lower powers of a repeated factor, and the numerators over the irreducible quadratics, have no root to isolate them, so finish those by comparing coefficients.

23 IMPROPER FRACTIONS B-23

Partial fractions apply only when the numerator has lower degree than the denominator. Otherwise divide first:

KEY

$$\frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$
$$\frac{x^2}{x^2 - 1} = 1 + \frac{1}{x^2 - 1}, \text{ and the second piece is row 3 of section 04, giving } x + \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C.$$

Check the degrees before writing any letters. Setting up partial fractions for an improper fraction produces a system with no solution, and the wasted minute is spent before anything looks wrong.

24 RATIONAL FORMS WITH A SHORTCUT B-24

- $\frac{1}{x(x^n + a)}$: multiply above and below by x^{n-1} , then put $t = x^n$.
- $\frac{x^2}{x^4 + a^4}$ is the half sum, and $\frac{1}{x^4 + a^4}$ is $\frac{1}{2a^4}$ times the difference, of $\frac{x^2 + a^2}{x^4 + a^4}$; each piece is section 13 run with $t = x \mp \frac{a}{x}$ and $x^2 + \frac{a^4}{x^2} = t^2 \pm 2a^2$.
- $\frac{1}{(x-a)(x-b)}$: split as $\frac{1}{a-b} \left(\frac{1}{x-a} - \frac{1}{x-b} \right)$ without solving anything.
- $\frac{x^{n-1}}{x^n + a}$: this is the f'/f pattern up to a constant.

These are the cases where partial fractions are slower than a substitution. A denominator built from a single power of x , or a symmetric quartic, is nearly always faster by one of these moves than by setting up unknown constants.

25 POWERS OF SINE AND COSINE B-25

$\int \sin^n x \cos^n x dx$	METHOD
n odd	keep one $\cos x$, put $t = \sin x$
m odd	keep one $\sin x$, put $t = \cos x$
both odd	either works; use the smaller power
both even and ≥ 0	drop the powers with multiple angles
$m + n$ even and negative	put $t = \tan x$

An odd power is a gift. It hands you the single factor that becomes dt , so the substitution is forced and the rest is a polynomial in t . Only the all even case needs identities, and only then is the work long.

26 DROPPING POWERS WITH MULTIPLE ANGLES B-26

KEY

- $\sin^2 x = \frac{1 - \cos 2x}{2}$ and $\cos^2 x = \frac{1 + \cos 2x}{2}$
- $\sin^3 x = \frac{3 \sin x - \sin 3x}{4}$ and $\cos^3 x = \frac{3 \cos x + \cos 3x}{4}$
- $\sin^4 x = \frac{3 - 4 \cos 2x + \cos 4x}{8}$

Raise the angle to drop the power. Each identity replaces a power by a first power of a multiplied angle, so a power of any size is flattened in a few steps and then integrated term by term. The first line is the one to remember; the others follow from it.

27 PRODUCTS OF SINES AND COSINES B-27

- $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$
- $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$
- $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$
- $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

$$\int \sin 3x \cos 5x dx = \frac{1}{8} \int (\sin 8x - \sin 2x) dx = -\frac{\cos 8x}{8} + \frac{\cos 2x}{8} + C.$$

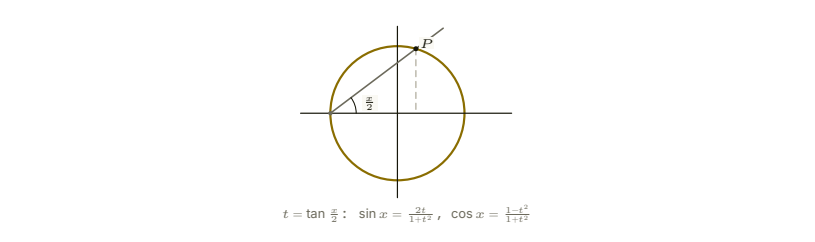
A product of different angles is never integrated as a product. Convert it to a sum first. The same four identities also handle $\sin ax \sin bx$ inside a larger integrand.

28 THE HALF ANGLE SUBSTITUTION B-28

For $\frac{1}{a + b \cos x}$, $\frac{1}{a + b \sin x}$ and $\frac{1}{a \sin x + b \cos x}$, put $t = \tan \frac{x}{2}$:

KEY

$$\sin x = \frac{2t}{1 + t^2}, \quad \cos x = \frac{1 - t^2}{1 + t^2}, \quad dx = \frac{2 dt}{1 + t^2}$$



It always works and it is rarely the fastest. The substitution turns any rational function of $\sin x$ and $\cos x$ into a rational function of t , so reach for it when nothing lighter fits. Sections 29 and 30 are the lighter options.

29 THE $\tan x$ SUBSTITUTION B-29

For $\frac{1}{a \sin^2 x + b \cos^2 x}$ and its relatives, divide above and below by $\cos^2 x$ and put $t = \tan x$:

KEY</