

CONCEPT MAP

DEFINITE INTEGRATION & AREA UNDER THE CURVE

5 TOPICS • 37 CORE IDEAS • ONE SHEET



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Ritesh Raj • M.Sc Mathematics, IIT Delhi
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1 THE INTEGRAL ITSELF 01-04

WHAT THE NUMBER IS
Two definitions, one number. If $F' = f$ on $[a, b]$ then

$$\int_a^b f(x) dx = F(b) - F(a) = [F(x)]_a^b$$

FUNDAMENTAL THEOREM OF CALCULUS

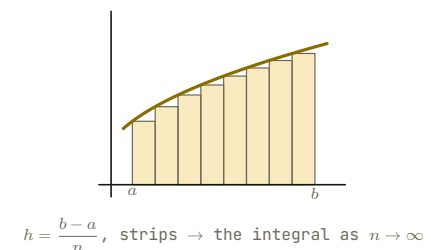
- First form:** if f is continuous and $g(x) = \int_a^x f(t) dt$, then $g'(x) = f(x)$. Integration and differentiation undo each other.
- Second form:** the displayed formula above; any one primitive will do, because two primitives differ by a constant that cancels.
- The constant of integration never appears in a definite integral.

The primitive must be valid on the whole interval. If f has a break inside $[a, b]$, as $\tan x$ does at $\pi/2$, the subtraction is meaningless and the integral must be split or declared improper first.

02 THE INTEGRAL AS A LIMIT OF A SUM A-02

KEY

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} h \sum_{r=1}^n f(a + rh), \quad h = \frac{b-a}{n}$$

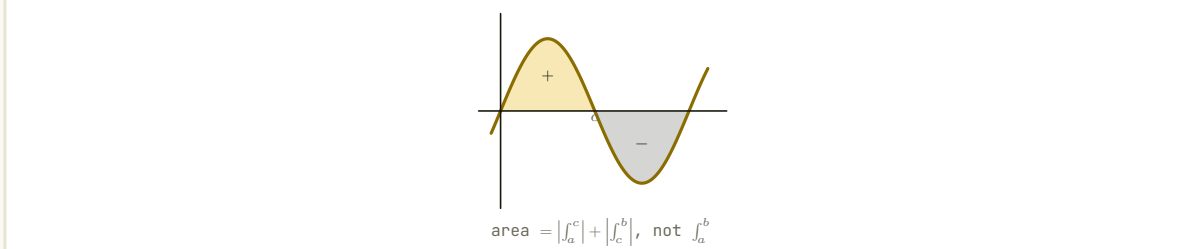


$h = \frac{b-a}{n}$, strips → the integral as $n \rightarrow \infty$

READING A LIMIT AS AN INTEGRAL

- Write the sum as $\frac{1}{n} \sum f(\frac{r}{n})$.
 - Replace $\frac{r}{n}$ by x and $\frac{1}{n}$ by dx .
 - The limits are the limits of $\frac{r}{n}$: if r runs from p to q with $\frac{p}{n} \rightarrow \alpha$ and $\frac{q}{n} \rightarrow \beta$, the integral runs from α to β .
- The limits come from the index, not the integrand.** A sum from $r = n$ to $3n$ becomes an integral from 1 to 3. Getting the range of r/n wrong, and misreading f from the summand, are the two things that go wrong in this family.

03 SIGNED AREA AND TRUE AREA A-03



- $f > 0$ on $[a, b]$: the integral is the area.
- $f < 0$ on $[a, b]$: the integral is negative and the area is its modulus.
- f changes sign at c : split there and add the magnitudes.

KEY

$$A = \int_a^c |f| dx + \int_c^b |f| dx$$

Solve $f(x) = 0$ before you integrate. A missed root lets positive and negative pieces cancel, and the uncanceled value is printed among the options. Nothing in the working looks wrong, which is exactly why it costs marks.

04 PROPERTIES I TO III A-04

- I. The variable is a dummy:** $\int_a^b f(x) dx = \int_a^b f(t) dt$.
 - II. Reversal:** $\int_a^b f = -\int_b^a f$, and $\int_a^a f = 0$.
 - III. Splitting:** $\int_a^b f = \int_a^c f + \int_c^b f$.
 - Linearity:** $\int_a^b (\lambda f + \mu g) = \lambda \int_a^b f + \mu \int_a^b g$.
- c need not lie between a and b .** Property III holds for any c for which all three integrals exist, because the signed convention of Property II absorbs the overshoot. This is what makes splitting at an outside point legal.

05 SUBSTITUTION: THE LIMITS MOVE TOO A-05

If $x = g(t)$ is one-one and differentiable on the relevant interval,

KEY

$$\int_a^b f(x) dx = \int_{g(a)}^{g(b)} f(g(t))g'(t) dt$$

- Change the limits at the moment you change the variable; there is then no need to substitute back.
- If the substitution is not one-one on the interval, split the interval first.
- A reversal of the limits is fine, because Property II takes care of the sign.

Never carry old limits into a new variable. Under $t = x^2$ the limits 0 and 1 happen to come back as 0 and 1; that is a coincidence of that one substitution, not a rule. Re-compute them every time.

06 INTEGRATION BY PARTS, DEFINITE FORM A-06

KEY

$$\int_a^b u v' dx = [uv]_a^b - \int_a^b u' v dx$$

- Choose u by **ILATE**: Inverse, logarithmic, algebraic, trigonometric, exponential.
- $\int e^x (f(x) + f'(x)) dx = e^x f(x) + C$, so the definite version is $[e^x f(x)]_a^b$.
- Repeated parts on $x^n e^x$ or $x^n \sin x$ terminates after n steps.

The boundary term is half the answer. In a definite integral the bracket is evaluated immediately and often vanishes at both ends, which is what makes a reduction formula collapse to a clean recurrence.

07 NEWTON-LEIBNITZ: DIFFERENTIATING AN INTEGRAL A-07

For f continuous and g, h differentiable,

KEY

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) dt = f(h(x))h'(x) - f(g(x))g'(x)$$

- You never evaluate the integral; you differentiate it.
- Constant limits give derivative zero, because the integral is a number.
- This turns an integral equation into a differential equation, which is how paragraph questions are built.

The lower limit carries a minus sign and its own chain factor. Dropping it is the standard slip. Check by taking g constant: the second term vanishes as it should.

08 THE INTEGRAL AS AN UNKNOWN A-08

Two situations look alike on the page, and telling them apart is the whole opening move.

THE INNER INTEGRAL HAS	SO IT IS
constant limits	a number: name it k , and do algebra
a variable limit	a function: differentiate it

$f(x) = e^x + \int_0^x (y + xe^y) f(y) dy$: both $\int_0^x y$ and $\int_0^x f$ are numbers, so $f(x) = e^x + \alpha + \beta xe^x$, and substituting back gives two equations in α, β .

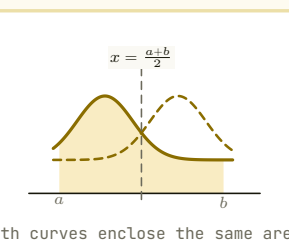
Name it before you touch it. Trying to evaluate an inner integral whose integrand still contains the unknown function is circular. Give it a letter, substitute the resulting form back, and solve the algebra.

B THE SYMMETRY PROPERTIES 09-15

09 THE KING PROPERTY 0-09

KEY

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$



both curves enclose the same area

The map $x \mapsto a+b-x$ sends a to b , sends b to a and fixes the midpoint: it reflects the interval in its own centre, and reflection does not change the enclosed area.

- On $[0, a]$ the reflection is $x \mapsto a-x$ on $[0, \pi/2]$ it swaps $\sin$ and $\cos$.
- King is worth using only when $f(x) + f(a+b-x)$ is simpler than either term.
- Adding the two forms and halving is the standard move: $2f = \int_a^b (f(x) + f(a+b-x)) dx$.

Three ways king goes wrong. Reflecting with the wrong sum $a+b$ applying it to one part of a split integral; and applying it when the sum does not simplify, which turns one integral into two.

10 KING I: THE CONSTANT SUM 0-10

If $f(x) + f(a+b-x) = k$ for every $x \in [a, b]$, then

KEY

$$\int_a^b f(x) dx = \frac{k(b-a)}{2}$$

$\int_0^{\pi/2} \sin^2 x dx$: here $\sin^2 x + \cos^2 x = 1$, so the value is $\frac{1-\pi/2}{2} = \frac{\pi}{4}$ with no antiderivative computed.

Look for the pair, not the function. The question is never "can I integrate this" but "what does this become under $x \mapsto a+b-x$, and does the sum collapse".

11 KING II: THE COMPLEMENTARY QUOTIENT 0-11

If $g(x) + g(a+b-x)$ never vanishes on $[a, b]$, then

KEY

$$\int_a^b \frac{g(x)}{g(x) + g(a+b-x)} dx = \frac{b-a}{2}$$

The reflection swaps the two terms of the denominator and swaps which sits on top, so the two forms add to 1 and King I applies with $k = 1$.

The answer never depends on g . Whatever ugly function is printed, if the integrand has this exact shape the value is half the length of the interval. Recognising the shape is the entire question.

12 KING III: THE $1 + e^x$ FAMILY 0-12

Let $c > 0$, $c \neq 1$, with $u(x) = 1 + e^{cx}$ and $v(x) = 1 + e^{-cx}$. Then

KEY

$$\int_a^b \frac{v(x)}{1 + c v(x)} dx = \frac{1}{c} \int_a^b v(x) dx$$

$$\int_{-\pi/2}^{\pi/2} \cos^2 x dx = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \cos^2 x dx + \frac{n-2}{n-1} \int_{-\pi/2}^{\pi/2} \sec^{n-2} x dx = \frac{\pi}{4}$$

The base is a decay. e^{-x} has vanished from the answer, so 2^x , e^{2x} and 2^{4x} all give the same value. The moment you see $1 + e^{6x}$ in a denominator and an even numerator on a symmetric interval, the factor is $\frac{1}{2}$. Check the numerator's parity too: with an odd numerator the halving is false.

13 THE INTERVAL $[0, 2a]$ 0-13

KEY

$$\int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_0^a f(2a-x) dx$$

Hence the two cases that get examined:

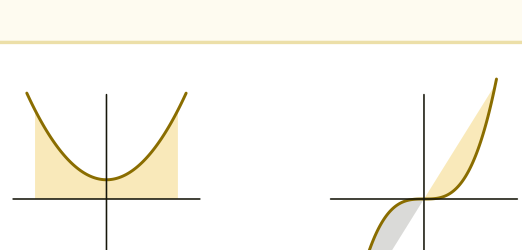
$$\int_0^{2a} f = \begin{cases} 0, & f(2a-x) = -f(x) \\ 2 \int_0^a f, & f(2a-x) = f(x) \end{cases}$$

What the two conditions mean. $f(2a-x) = f(x)$ says the graph is symmetric about the line $x = a$; $f(2a-x) = -f(x)$ says it has a half-turn centre at $(a, 0)$. Test the condition, do not assume it from the shape of the formula.

14 ODD AND EVEN 0-14

KEY

$$\int_{-a}^a f(x) dx = \begin{cases} 2 \int_0^a f(x) dx, & f \text{ even} \\ 0, & f \text{ odd} \end{cases}$$

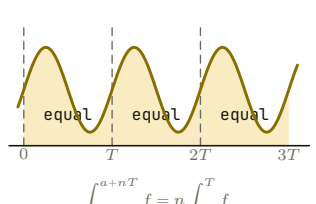


- This is King with $a+b = 0$: the reflection is $x \mapsto -x$.
 - Split a general integrand into its even and odd parts; only the even part survives.
 - odd $\times$ odd = even; odd $\times$ even = odd.
- Check the parity, do not guess it.** $x \sin x$ is even, $x^3 \sin x$ is odd, and $\sin(x^2)$ is even. Substitute $-x$ once rather than trusting the look of the expression.

15 PERIODIC INTEGRANDS 0-15

If f has period T , then for all real a, b and integers m, n :

- $\int_a^{a+T} f = \int_c^{c+T} f$, so the starting point does not matter.
- $\int_a^{a+nT} f = n \int_a^{a+T} f$.
- $\int_{a+T}^{a+2T} f = (n-m) \int_a^{a+T} f$.
- $\int_{-a+T}^{-a} f = \int_a^T f$.



$\int_c^{c+T} f = \int_a^T f$

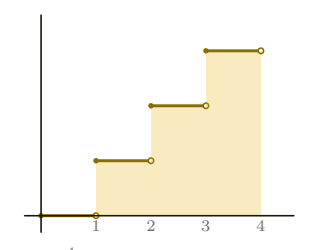
Find the period of the integrand, not of the parts. $|\sin x|$ has period π , not 2π ; $\sin^2 x$ has period π ; $\{x\}$ has period 1. Peel off whole periods first, then integrate the short remainder.

16 EVALUATION & ESTIMATION 16-24

16 WHEN THE INTEGRAND BREAKS C-16

Every property so far assumed one formula on the whole interval. When that fails, the repair is always Property III: split at every point where the formula changes.

INTEGRAND	SPLIT AT
$ g(x) $	every root of g inside the interval
$ g(x) $ or $ g(x) ^2$	every x where g takes an integer value
a piecewise rule	every changeover point in the definition



$\int_0^1 n(x-0.5)^{n-1} dx = 1 - 0.5^n$

The split points come from the inside. For $\int_0^1 |x \sin x| dx$ they are not $-1, 0, 1$; they are where $x \sin x$ crosses an integer. A step-function integral is always a finite sum of rectangles.

17 REDUCTION FORMULAE C-17

Integrating by parts once turns a power into a recurrence:

KEY

$$I_n = \int_a^b x^{n-2} \sin^2 x dx \Rightarrow I_n = \frac{n-1}{n} I_{n-2}$$

- $\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$.
- $\int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx$.
- Iterate down to $I_0 = \frac{\pi}{2}$ or $I_1 = 1$ for the sine and cosine families.

A recurrence is an answer. Questions in this family usually ask for a relation between I_n and I_{n-2} , or for I_n at one specific n ; either way you never need the general closed form, only the step down.

18 WALLIS' FORMULA C-18

For an integer $n \geq 2$, the chain of fractions runs down to $\frac{1}{2}$ (n even) or $\frac{3}{2}$ (n odd) and then stops:

KEY

$$\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \lambda$$

with $\lambda = \frac{\pi}{2}$ for n even and $\lambda = 1$ for n odd.

THE MIXED FORM

$$\int_0^{\pi/2} \sin^{m-1} x \cos^{n-1} x dx = \frac{(m-1)(m-3) \cdots (n-1)(n-3) \cdots \lambda}{(m+n)(m+n-2) \cdots \lambda}$$

Here every chain runs down to 2 or 1, and $\lambda = \frac{\pi}{2}$ when m and n are both even, $\lambda = 1$ otherwise. $m = n = 2$: $\frac{1 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = \frac{\pi}{16}$; $m = 3, n = 2$: $\frac{2 \cdot 1}{5 \cdot 3 \cdot 1} \cdot 1 = \frac{2}{15}$.

The extra $\pi/2$ appears only when both powers are even. Note also that the two chains stop at different places in the two forms: the single-power chain halts one fraction early, the mixed chains run all the way down. Mixing the two conventions is what makes this formula misfire. The equality of $\sin^n$ and $\cos^n$ is the king property on $[0, \pi/2]$, not Wallis.

19 THE GAMMA FUNCTION C-19

KEY

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$

- $\Gamma(n+1) = n\Gamma(n)$, by one integration by parts.
 - $\Gamma(1) = 1$, hence $\Gamma(n) = (n-1)!$ for every positive integer n .
 - Cauchy-Schwarz follows from $\int (f + \lambda g)^2 \geq 0$ having non-positive discriminant in λ .
 - The modulus bound is stated for $a < b$, so reverse the limits first if they arrive the other way round; Cauchy-Schwarz is unaffected by the order.
- Improper integrals.** When a limit is infinite or the integrand blows up inside, define the integral as a limit and check that the limit is finite. $\int_a^\infty \frac{dx}{x^p}$ converges exactly when $p > 1$; $\int_0^1 \frac{dx}{x^p}$ converges exactly when $p < 1$.

20 THE BETA FUNCTION C-20

KEY

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

- Symmetry:** $B(m, n) = B(n, m)$, which is the king property on $[0, 1]$, where the reflection is $x \mapsto 1-x$.
- For positive integers, $B(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$.
- $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = B(m, n)$.
- $\int_0^1 x^{m-1} (a-x)^{n-1} dx = a^{m+n-1} B(m, n)$.

Recognising a Beta integral. Any integral over $[0, 1]$ of a power of x times a power of $1-x$ is a Beta value. If the bracket is $1-x^2$ rather than $1-x$, substitute $t = x^2$ first; that is the only move the recent papers have needed.

21 BETA, GAMMA AND WALLIS TOGETHER C-21

KEY

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

TRIGONOMETRIC FORM

$$\int_0^{\pi/2} \sin^2 x \cos^2 x dx = \frac{1}{2} B(\frac{3}{2}, \frac{3}{2}) = \frac{\pi}{16}$$

- Substituting $x = \sin^2 \theta$ in B is what produces that form.
- Wallis' formula in section 18 is this identity written out with factorials.
- $B(\frac{1}{2}, \frac{1}{2}) = \pi$, which is $\Gamma(\frac{1}{2})^2$ and gives $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Advanced only, and worth ten seconds. JEE Main will not name Beta or Gamma, but Advanced papers print integrals in exactly these shapes. If an integrand is a power of x against a power of $1-x$, or $\sin^n \cos^n$ on $[0, \pi/2]$, this line answers it without any integration.

22 INVERSE-FUNCTION PAIRING C-22

If f is continuous and strictly increasing on $[a, b]$, then

KEY

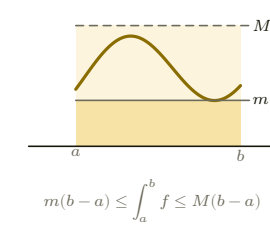
$$\int_a^b f(x) dx + \int_{f(a)}^{f(b)} f^{-1}(y) dy = b f(b) - a f(a)$$

The two integrals are the areas on the two sides of the curve inside the rectangle $[0, b] \times [0, f(b)]$, less the corner rectangle $[0, a] \times [0, f(a)]$. The picture assumes the curve sits in the first quadrant; the identity does not.

The giveaway is a sum of two integrals you cannot do. Check whether the second integrand is the inverse of the first and whether its limits are the images of the first pair. If both hold, the answer is a single subtraction of products.

23 ESTIMATING WITHOUT EVALUATING C-23

- Comparison:** $g \leq f \leq h$ on $[a, b]$ with $a < b$ $\Rightarrow \int_a^b g \leq \int_a^b f \leq \int_a^b h$.
- Global bounds:** $m \leq f \leq M$ on $[a, b]$ with $a < b$ $\Rightarrow m(b-a) \leq \int_a^b f \leq M(b-a)$.



$m(b-a) \leq \int_a^b f \leq M(b-a)$

- Bound the integrand, not the integral.
 - Integrate the two constant bounds over the same interval.
 - If the bound is still too loose, split the interval and bound each piece separately.
- Some questions never want the number.** When the answer choices are intervals rather than values, evaluating is a trap: the integral may have no elementary primitive at all, and the bounds are the whole solution.

24 TWO MORE INEQUALITIES C-24

KEY

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx \quad (a < b)$$

CAUCHY-SCHWARZ FOR INTEGRALS

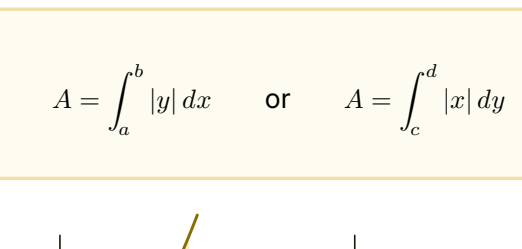
- The modulus bound follows from comparison applied to $-|f| \leq f \leq |f|$.
 - Cauchy-Schwarz follows from $\int (f + \lambda g)^2 \geq 0$ having non-positive discriminant in λ .
 - The modulus bound is stated for $a < b$, so reverse the limits first if they arrive the other way round; Cauchy-Schwarz is unaffected by the order.
- Improper integrals.** When a limit is infinite or the integrand blows up inside, define the integral as a limit and check that the limit is finite. $\int_a^\infty \frac{dx}{x^p}$ converges exactly when $p > 1$; $\int_0^1 \frac{dx}{x^p}$ converges exactly when $p < 1$.

25 AREA BOUNDED BY CURVES 25-34

25 AREA, AND WHICH WAY TO CUT IT 0-25

KEY

$$A = \int_a^b |g| dx \quad \text{or} \quad A = \int_c^d |x| dy$$



vertical strip: $A = \int_a^b g dx$
horizontal strip: $A = \int_c^d x dy$

- Vertical strips when the region has a clear ceiling and floor as functions of x .
- Horizontal strips when it has a clear left and right as functions of y : a sideways parabola, or a region whose top is made of two different curves.
- Choosing the wrong direction turns one integral into three.

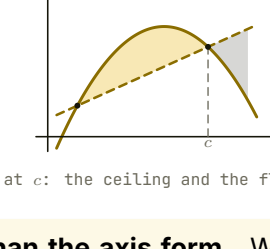
Three questions before any integration. What is the region? What are its limits? Which curve is on top? A sketch answers all three in less time than one wrong integral costs.

26 CURVES THAT CROSS 0-26

If $g \geq f$ on all of $[a, b]$ the area is $\int_a^b (g-f) dx$. If they cross at c inside, the roles swap:

KEY

$$A = \int_a^c (g-f) dx + \int_c^b (f-g) dx$$



split at c : the ceiling and the floor swap

The difference form is safer than the axis form. Writing $\int (g-f)$ needs no sign discussion about where the curves sit relative to the x -axis, only about which of the two is higher. Solve $f = g$ first and split at every root.

27 THE SIX STANDARD CONFIGURATIONS 0-27

- Curve $y = f(x) \geq 0$, the x -axis and $x = a, x = b$: $A = \int_a^b f$.
- $g \geq f$ on all of $[a, b]$: $A = \int_a^b (g-f)$.
- Two curves meeting at $x = a$ and $x = b$: same formula, with a, b the roots of $f = g$.
- Curve crossing the axis at c , positive then negative: $A = \int_a^c f - \int_c^b f$.
- Two curves crossing at c