

CONCEPT MAP

DIFFERENTIAL EQUATIONS

5 TOPICS • 36 CORE IDEAS • ONE SHEET



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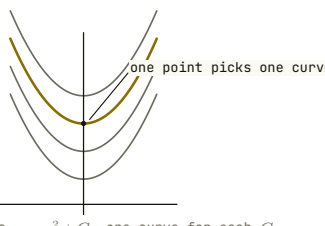
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1 BASICS & FORMATION 01-06

01 WHAT A DIFFERENTIAL EQUATION IS A-01

An equation relating an unknown function to one or more of its derivatives. If the unknown depends on a single variable the equation is **ordinary**, and only ordinary equations are on the syllabus.

$$2x \frac{dy}{dx} - 3y = 5, \quad 4 \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$$



$y' = 2x$ gives $y = x^2 + C$, one curve for each C

A derivative must actually appear. $x^2 + 2x + y^2 = 0$ is a curve, not a differential equation. And solving one almost never gives a single answer: $y' = 2x$ gives $y = x^2 + C$, a whole family, and one extra condition picks out one member of it.

02 ORDER AND DEGREE A-02

- Order:** the order of the highest derivative present.
- Degree:** the power of that highest derivative, **after** the equation is made polynomial in every derivative, free of radicals, fractional powers and derivatives in a denominator.
- If no such polynomial form exists, the degree is not defined.
- An equation containing $\sin y'$ or $e^{y'}$ has no degree.

$\sqrt[n]{y''} = \sqrt[n]{y'' + 5}$: raise to the n th power to get $(y'')^n = (y'' + 5)^n$, so the order is 2 and the degree is 4.

Clear the radicals first, and only then read the degree. Degree is a statement about the polynomial in the derivatives, so reading it off before rationalising is the single commonest error here, and examiners set the trap deliberately.

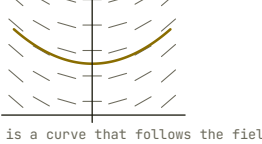
03 SOLUTIONS AND HOW MANY CONSTANTS A-03

- General solution:** contains as many independent arbitrary constants as the order of the equation.
- Particular solution:** one member, fixed by initial or boundary conditions.
- Singular solution:** a solution that no value of the constant produces.
- To verify a proposed solution, differentiate it the right number of times and substitute back.

Count the constants before anything else. An n th order equation that can be solved for its highest derivative has exactly n independent constants in its general solution, so a printed general solution carrying the wrong number is wrong. Singular solutions carry none and are counted separately. That same count runs in reverse in section 05.

04 THE GEOMETRIC MEANING A-04

A first order equation $\frac{dy}{dx} = F(x, y)$ prescribes a **direction** at every point where F is defined. The solutions are the curves that follow those directions.



a solution is a curve that follows the field

- Where F and $\partial F / \partial y$ are continuous, exactly one solution curve passes through each point.
- The general solution is the whole family; an initial condition names the point the curve must pass through.
- Two such curves cannot cross. Where that fails, at a root of $g(y)$ in a separable equation or at a singular solution, several curves can pass through one point.

Read the picture when the algebra stalls. Questions that ask which of four sketched families can solve a given equation are answered by testing the sign of y' in each quadrant, not by solving anything.

05 FORMATION: ELIMINATING THE CONSTANTS A-05

- Count the arbitrary constants in the family; call the number n .
- Differentiate the relation n times with respect to x .
- Eliminate the n constants using the $n+1$ equations now available.

The result has order **exactly** n : eliminating n unknowns from $n+1$ equations leaves one relation, and the highest derivative it can contain is the n th. The equation formed may pick up extra singular solutions belonging to no member of the family.

$y = Ae^{2x} + Be^{-x}$ has two constants, so differentiate twice: $y'' = Ae^{2x} + Be^{-x} = y$, giving $y'' - y = 0$.

Only independent constants count. $y = Ae^{x+1}$ looks like two, but Ae^B is a single constant, so the equation formed is first order. Reduce the family to its genuinely independent parameters before differentiating.

06 WHICH FAMILY, WHICH ORDER A-06

FAMILY	ORDER OF ITS EQUATION
all lines through the origin	1
all non-vertical lines	2
all circles centred on the x -axis	2
all circles of fixed radius r	2
all circles in the plane	3
all parabolas $y^2 = 4ax$	1
all central conics $ax^2 + by^2 = 1$	2
all conics with axes parallel to the axes	4

Count the freedoms in the picture. A circle in the plane needs a centre and a radius, which is three numbers, so its differential equation is third order. Reading the geometry is faster than writing the general equation and differentiating.

2 SEPARABLE & EXACT 07-13

07 VARIABLE SEPARABLE B-07

If the equation can be written with all the x on one side and all the y on the other,

KEY

$$\frac{dy}{dx} = f(x)g(y) \Rightarrow \int \frac{dy}{g(y)} = \int f(x)dx + C$$

- Separate, then integrate both sides.
- Any arbitrary constant is enough; do not write one on each side.
- Dividing by $g(y)$ loses the constant solutions $g(y) = 0$, so check them separately.

The lost solutions are worth a mark. Dividing by $g(y)$ silently assumes $g(y) \neq 0$. Each root of g gives a constant function that solves the equation, and a multiple-correct question will list one of them among the options.

08 REDUCIBLE TO SEPARABLE B-08

If the right side depends on x and y only through a linear block,

KEY

$$\frac{dy}{dx} = f(ax + by + c) \Rightarrow t = ax + by + c \quad (b \neq 0)$$

Then $\frac{dt}{dx} = a + b \frac{dy}{dx} = a + b f(t)$, which separates at once.

$\frac{dy}{dx} = (x + y)^2$: put $t = x + y$, so $\frac{dt}{dx} = 1 + t^2$ and $\tan^{-1} t = x + C$.

Look for the repeated block. Whenever the same combination $ax + by$ appears more than once, substituting for it is almost always the intended move, and it converts the equation to separable form in one line.

09 OTHER REDUCTIONS TO SEPARABLE B-09

- $\frac{dy}{dx} = f_1(x)g_1(y) + f_2(x)g_2(y)$ separates only if the right side happens to factorise, as $xy + x = x(y+1)$ does. If it does not, the equation is usually linear instead.
- $y' = f(x/g(y))$ hidden behind a common factor: cancel it, then check whether the cancelled factor is itself a solution.
- $\frac{dy}{dx} + \frac{y}{x} = 0$ and its relatives separate directly, giving $xy = C$.
- An equation in $\frac{dx}{dy}$ may separate when the same equation in $\frac{dy}{dx}$ does not.

Try turning the equation upside down. Treating x as the unknown and y as the variable costs nothing and rescues a surprising number of equations, both here and for the linear form in section 20.

10 EXACT DIFFERENTIALS TO KNOW BY SIGHT B-10

KEY

$$d(xy) = x dy + y dx, \quad d\left(\frac{y}{x}\right) = \frac{x dy - y dx}{x^2}$$

$$d\left(\ln \frac{y}{x}\right) = \frac{x dy - y dx}{xy}, \quad d\left(\tan^{-1} \frac{y}{x}\right) = \frac{x dy - y dx}{x^2 + y^2}$$

$$x dx + y dy = \frac{1}{2} d(x^2 + y^2), \quad d\left(\frac{-1}{xy}\right) = \frac{x dy + y dx}{x^2 y^2}$$

The fastest method in the chapter when it applies. It is also the slowest to find if the list is not memorised, because nothing in the equation announces it. Learn these by sight and scan for them before choosing any other method.

11 THE TWO SIGNALS B-11

SEEING	DIVIDE BY
$x dy + y dx$	nothing: it is $d(xy)$ already
$x dy - y dx$ with x^2	x^2 , giving $d(y/x)$
$x dy - y dx$ with y^2	y^2 , giving $-d(x/y)$
$x dy - y dx$ with xy	xy , giving $d(\ln \frac{y}{x})$
$x dy - y dx$ with $x^2 + y^2$	$x^2 + y^2$, giving $d(\tan^{-1} \frac{y}{x})$

The rest of the equation chooses the divisor. All four divisors are legal; pick the one that leaves the remaining terms expressible in the same new variable. That is the whole skill in this section.

12 THE EXACTNESS TEST B-12

$M(x, y)dx + N(x, y)dy = 0$ is **exact** when

KEY

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- If the test passes on a rectangle, there is a function u there with $du = M dx + N dy$, and the solution is $u = C$.
- Find u by integrating M with respect to x , treating y as constant.
- The constant of that integration is a function of y ; fix it by differentiating with respect to y and comparing with N .

Test before you hunt. The partial derivative check takes ten seconds and tells you whether the inspection method of section 11 has any chance. If the test fails, the equation may still become exact after multiplying by an integrating factor.

13 INTEGRATING FACTORS BY INSPECTION B-13

A non-exact equation can often be made exact by multiplying through by a factor. The ones worth trying:

- $\frac{1}{x^2}$, $\frac{1}{y^2}$, $\frac{1}{xy}$ or $\frac{1}{x^2 + y^2}$, matching the table in section 11.
- $x^m y^n$ for suitable m, n , found by comparing powers after multiplying.
- If $\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$ is a function of x alone, its exponential integral is an integrating factor.
- If $\frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$ is a function of y alone, the same construction works in y .

The linear equation of section 18 is this idea in its commonest case. There the integrating factor is $e^{\int P dx}$, and it is exactly the factor that turns the left side into the derivative of a product.

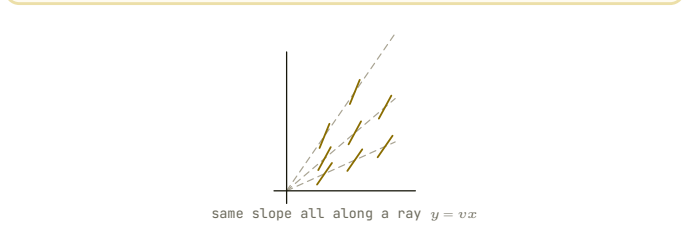
3 HOMOGENEOUS & LINEAR 14-21

14 HOMOGENEOUS EQUATIONS C-14

$f(x, y)$ is **homogeneous of degree** n if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$. $M dx + N dy = 0$ is homogeneous when M and N are homogeneous of the **same** degree, which is how most questions present it. Equivalently the equation can be written

KEY

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$



The picture is the definition. The slope depends only on y/x , so it is constant along every ray through the origin. That is what makes $y = vx$ the right substitution, and it is also how to spot the type at a glance.

15 THE $y = vx$ SUBSTITUTION C-15

- Put $y = vx$, so $\frac{dy}{dx} = v + x \frac{dv}{dx}$.
- The equation becomes $v + x \frac{dv}{dx} = F(v)$.
- Separate: $\frac{dv}{F(v) - v} = \frac{dx}{x}$, valid where $F(v) \neq v$. Each root v_0 of $F(v) = v$ gives an extra straight-line solution $y = v_0 x$.
- Integrate, then put $v = \frac{y}{x}$ back.

Substitute $x = vx$ instead when the equation is simpler in x/y . The two are equally valid; choose whichever makes the leftover integral easier. Forgetting to substitute back at the end is the usual lost mark.

16 REDUCIBLE TO HOMOGENEOUS C-16

For $\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$ with $a_1 b_2 \neq a_2 b_1$, shift the origin to where the two lines meet:

KEY

$$x = X + h, \quad y = Y + k, \quad \begin{aligned} a_1 h + b_1 k + c_1 &= 0 \\ a_2 h + b_2 k + c_2 &= 0 \end{aligned}$$

The shift leaves $\frac{dY}{dX} = \frac{dy}{dx}$ and kills both constants, giving a homogeneous equation in X and Y .

The condition is exactly that the two lines meet. The pair of linear equations for h and k has a unique solution when $a_1 b_2 - a_2 b_1 \neq 0$, which is when the lines are not parallel. If they are, section 17 applies instead.

17 THE PARALLEL CASE C-17

If $a_1 b_2 = a_2 b_1$, the two lines are parallel or coincident, so there is no unique point of intersection and the shift fails. The block $a_1 x + b_1 y$ then appears in both, so put

KEY

$$z = a_1 x + b_1 y$$

which reduces the equation to separable form, as in section 08. $(2x + 3y - 2)dx + (4x + 6y - 7)dy = 0$: since $4x + 6y = 2(2x + 3y)$, put $z = 2x + 3y$.

Check the ratio of the coefficients first. Comparing a_1/a_2 with b_1/b_2 takes a moment and decides between section 16 and this one. Attempting the shift on such a pair gives a system with no solution, or with infinitely many, and in neither case a usable (h, k) .

18 LINEAR FIRST ORDER C-18

KEY

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$IF = e^{\int P dx}, \quad y \cdot IF = \int Q \cdot IF dx + C$$



Multiplying by the integrating factor turns the left side into $\frac{d}{dx}(y \cdot IF)$, so the equation integrates directly.

Get the coefficient of y' to 1 first. P is read off only after dividing through, and reading it from an equation still carrying a coefficient on y' produces the wrong integrating factor and a wrong answer that looks plausible.

19 USING THE INTEGRATING FACTOR WELL C-19

- Simplify before multiplying: $e^{bf} I = f$ when $f' > 0$, so keep track of the sign on each interval.
- No constant is needed inside $\int P dx$; adding one multiplies the factor by a number that cancels.
- $IF = e^{\int P dx}$ is never zero on an interval where P is continuous, so multiplying by it loses no solutions there.
- After integrating, divide by the factor once and only once.

$P = \frac{2}{x}$ gives $IF = e^{\int 2/x dx} = x^2$, and the equation becomes $\frac{d}{dx}(x^2 y) = x^2 Q$.

This is where the arithmetic goes wrong. Not in the method, which is mechanical, but in simplifying the exponential. Write the integrating factor in its simplest form on its own line before it multiplies anything.

20 LINEAR IN x INSTEAD C-20

An equation that is not linear in y may be linear in x :

KEY

$$\frac{dx}{dy} + P(y)x = Q(y), \quad IF = e^{\int P dy}$$

$(y^2 - x) \frac{dy}{dx} = 1$ is not linear in y , but inverting gives $\frac{dx}{dy} + x = y^2$, which is.

Look at the powers before deciding. If y appears to powers other than the first but x appears only linearly, turn the equation over. The test costs one glance and rescues the whole question.

21 BERNOULLI C-21

KEY

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \quad n \neq 0, 1$$

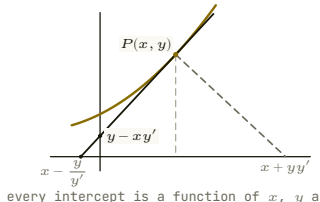
- Divide throughout by y^n , noting that this discards $y = 0$ when $n > 0$; check it separately.
- Put $v = y^{1-n}$, so that $\frac{dv}{dx} = (1-n)y^{-n} \frac{dy}{dx}$.
- The equation becomes $\frac{dv}{dx} + (1-n)Pv = (1-n)Q$, which is linear in v .
- Solve, then put $v = y^{1-n}$ back.

The excluded cases need no work. $n = 0$ is already linear and $n = 1$ is separable. Read n off the right side after collecting every y that is not to the first power there; if two different powers of y survive, the equation is not of this type.

4 CURVES & APPLICATIONS 22-29

22 THE GEOMETRY DICTIONARY D-22

AT $P(x, y)$ WITH SLOPE y'	EXPRESSION
tangent at P	$Y - y = y'(X - x)$
normal at P	$Y - y = -\frac{1}{y'}(X - x)$
x -intercept of the tangent	$\frac{x - y}{y'}$
y -intercept of the tangent	$\frac{y - xy'}{y'}$
length of the sub-tangent	$\left \frac{y}{y'} \right $
length of the sub-normal	$ yy' $
signed area from a to x	$\int_a^x f(t) dt$



every intercept is a function of x, y and y'

Use X, Y for the running point. Writing the tangent as $y - y = y'(x - x)$ says nothing. Keep the point of contact as (x, y) and the moving point as (X, Y) ; then setting $Y = 0$ or $X = 0$ gives the intercepts clearly, and each is a function of x, y and y' , which is a differential equation waiting to be written down.

23 CURVES FROM A TANGENT CONDITION D-23

- Translate the stated condition into the dictionary of section 22.
- The result is a relation between x, y and y' , which is the differential equation.
- Solve it by whichever of the earlier methods fits.
- Use the given point to fix the constant.

"The y -intercept of the tangent equals the cube of the abscissa" becomes $y - xy' = x^3$, that is $y' - \frac{y}{x} = -x^2$, linear with $IF = 1/x$.

Translate, do not solve, in the first step. Most of the marks here are lost in the sentence, not the calculus: read exactly which intercept, which length, and which point the condition is about before writing a single derivative.

24 CURVES FROM AN AREA CONDITION D-24

A condition on the area under the curve becomes an equation on differentiating, by the fundamental theorem:

KEY

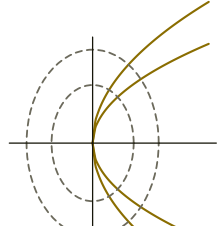
$$\int_a^x f(t) dt = g(x) \Rightarrow f(x) = g'(x)$$

- Differentiate once for each integral sign present.
- Fix the constant by putting $x = a$ in the original relation, where the integral vanishes.
- The same move handles conditions on arc length or on a volume of revolution.

The check is part of the method. Differentiating can introduce solutions the original relation does not have, so substitute the answer back into the integral form before quoting it.

25 ORTHOGONAL TRAJECTORIES D-25

- Form the differential equation of the given family, eliminating its constant, to get $y' = F(x, y)$.
- Replace y' by $-1/y'$, giving $y' = -\frac{1}{F(x, y)}$.
- Solve the new equation.



$y^2 = 4ax$ and $2x^2 + y^2 = c$ meet at right angles

Eliminating the constant is what makes it work. After step one the slope is a function of position alone, so it describes whichever member passes through the point. Only then does replacing the slope by its negative reciprocal describe the perpendicular curves.

26 GROWTH AND DECAY D-26

KEY

$$\frac{dN}{dt} = kN \Rightarrow N(t) = N_0 e^{kt}$$

- $k > 0$ is growth, $k < 0$ is decay, and $N_0 = N(0)$.
- Half-life = $\frac{\ln 2}{|k|}$ for a decaying quantity.
- Continuously compounded interest, bacterial growth and radioactive decay are the same equation.

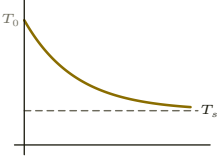
Two readings fix everything, and you rarely need k . One reading gives N_0 and a second gives e^{kt_1} . Every later value is a power of that number: if the quantity halves in 15 minutes then after 45 minutes the factor is $(\frac{1}{2})^3$. Solving for k wastes time and invites slips.

27 NEWTON'S LAW OF COOLING D-27

KEY

$$\frac{dT}{dt} = -k(T - T_s), \quad k > 0$$

$$T(t) = T_s + (T_0 - T_s)e^{-kt}$$



It is growth and decay in disguise. Putting $u = T - T_s$ turns it into $u' = -ku$, because T_s is a constant. The temperature approaches the surroundings but never reaches them, which is the shape the question usually tests.

28 MIXING PROBLEMS D-28

For a tank holding a dissolved quantity $x(t)$:

KEY

$$\frac{dx}{dt} = (\text{rate in}) - (\text{rate out})$$

- Rate in = inflow concentration $\times$ inflow rate.
- Rate out = $\frac{x}{V(t)} \times$ outflow rate, assuming the tank is kept well stirred.
- $V(t) = V_0 + (\text{inflow} - \text{outflow})t$, which is constant only when the two rates are equal.
- The result is linear in x , so use the integrating factor.

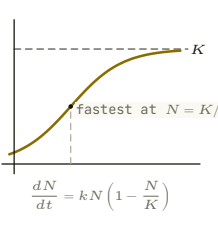
Track the volume separately. If the inflow and outflow rates differ, V changes with time and the outflow term is not proportional to x alone. Writing V down first prevents the commonest error in this family.

29 BOUNDED GROWTH D-29

When growth is limited by a carrying capacity K :

KEY

$$\frac{dN}{dt} = kN \left(1 - \frac{N}{K}\right)$$



- The equation is separable, and partial fractions give the solution.
- $N = 0$ and $N = K$ are the constant solutions.
- For $0 < N < K$ growth is fastest at $N = K/2$, the inflection point of the curve; above K the population falls.

Population dynamics is on the Advanced syllabus, not the Main one. What is asked is usually the shape, the constant solutions or the point of fastest growth, rather than the full solution, so those three facts are the ones to carry.

5 HARDER FORMS & METHOD 30-36

30 HIDDEN INSIDE A LIMIT E-30

A limit of the $\frac{0}{0}$ form as $t \rightarrow x$ is a derivative in disguise.