

CONCEPT MAP

BASIC ALGEBRA & LOGARITHMS

5 TOPICS • 38 CORE IDEAS • ONE SHEET



INTERVALS & SIGN

B1-B9

81

THE NUMBER SYSTEMS

A-B1

- $NCWCZCQCRC$.
- A rational is p/q with $q \neq 0$; its decimal terminates or recurs.
- An irrational has a non-terminating, non-recurring decimal, and $\sqrt{2}$, e and π are the standard examples.
- Between any two distinct reals there lies a rational and an irrational, so neither set has gaps you can point at.

Rational plus irrational is always irrational. So is rational times irrational, provided the rational is not zero. But irrational plus irrational can be anything: $\sqrt{2} + (-\sqrt{2}) = 0$. That asymmetry is what the questions are built on.

82

INTERVALS AND THEIR TYPES

A-B2

INTERVAL	MEANING
(a, b)	open: $a < x < b$, neither end included
$[a, b]$	closed: $a \leq x \leq b$, both ends included
$[a, b)$	half open: $a \leq x < b$
$(a, b]$	half open: $a < x \leq b$
(a, ∞)	unbounded above; ∞ is never included
$(-\infty, b]$	unbounded below, right end included
$(-\infty, \infty)$	the whole of $\mathbb{R}$
$[a, a]$	degenerate: the single point $\{a\}$
(a, a)	empty

83

WORKING WITH INTERVALS

A-B3

- Union of overlapping intervals is one interval, and so is the union of two abutting intervals provided the shared endpoint belongs to one of them, as in $[1, 2] \cup [2, 5] = [1, 5]$. If it belongs to neither, the union stays in two pieces.
- Intersection takes the larger left end and the smaller right end, and is empty if they cross.
- A neighbourhood of a is $(a-\delta, a+\delta)$; a deleted neighbourhood removes a itself.
- Excluding a point from an interval splits it: $[0, 5] \setminus \{2\} = [0, 2) \cup (2, 5]$.

$[1, 4] \cap [2, 7] = (2, 4)$, while $[1, 2] \cup (2, 5]$ cannot be written as one interval.

Report the answer as a union of intervals, in increasing order. A set of solutions is not a list of inequalities; writing it as intervals makes the boundary decisions explicit and is what the printed options are matched against.

84

THE SIGN OF A SINGLE FACTOR

A-B4

- $x-a$ is negative for $x < a$, zero at a , positive for $x > a$.
- $a-x$ is the opposite, so make every factor of the form $x-a$ before starting.
- A factor with a negative coefficient of x flips the whole sign scheme; take the minus outside instead.
- $(x-a)^2$ is never negative, and $(x-a)^{2k}$ likewise.

Normalise every factor first. Write $3-x$ as $-(x-3)$ and pull the minus signs out to the front. The sign scheme then depends only on the roots, and one leading sign fixes the whole picture.

85

THE WAVY CURVE METHOD

A-B5

1. Bring everything to one side so the inequality reads $f(x) > 0$, < 0 , ≥ 0 or ≤ 0 against zero.
- ii. Factorise completely into linear and irreducible quadratic factors.
- iii. Make the coefficient of x in every linear factor positive, then make the **overall leading coefficient** positive: if an odd number of minus signs came out, multiply through by -1 and reverse the inequality.
- iv. Mark all the roots on a number line in increasing order.
- v. Start above the line to the far right and draw a wave alternating through each simple root.
- vi. Read off the intervals where the wave is on the required side.

Start on the right and above, once the leading coefficient is positive. The sign at the far right is the sign of the leading coefficient, not of the number of factors. Pull every minus sign out and clear it by flipping the inequality; only then is the far right the $+$ region.

86

REPEATED FACTORS AND THE EVEN-POWER RULE

A-B6

- An odd power of a factor changes the sign at that root; the wave crosses.
- An even power does not; the wave touches the axis and comes back.
- An irreducible quadratic written with positive leading coefficient and $D < 0$ is always positive, so it divides out without changing the inequality.
- A repeated root is still a root, so it must appear on the line even though the sign does not change there.

The same rule reappears three times in the course. Here for inequalities, on the Application of Derivatives sheet for the sign of f' , and again for points of inflection with f'' . Learn it once as a statement about parity of exponents.

87

RATIONAL INEQUALITIES

A-B7

For $\frac{P(x)}{Q(x)} > 0$, do not cross-multiply. Instead:

1. Bring everything to one side over a single denominator.
- ii. Apply the wavy curve method to the product $P(x)Q(x)$, since a quotient and a product have the same sign.
- iii. Mark the roots of Q with hollow dots: they are never in the answer, whatever the inequality sign.
- iv. For $>$ or $\leq$, fill in the roots of P but leave the roots of Q hollow; a root shared by both stays hollow.

$\frac{x-1}{x-3} \geq 0$ gives $(x-1)(x-3) \geq 0$ with $x \neq 3$, so x is $(-\infty, 1] \cup (3, \infty)$.

Cross-multiplying assumes the denominator is positive. When it is negative the inequality flips, and multiplying without knowing its sign is the single commonest error in this topic.

88

THE SIGN OF A QUADRATIC

A-B8

For $f(x) = ax^2 + bx + c$ with $a > 0$ and $D = b^2 - 4ac$:

- $D < 0$: $f(x) > 0$ for every real x .
- $D = 0$: $f(x) \geq 0$ always, zero only at $x = -b/2a$.
- $D > 0$: f is negative strictly between the roots and positive outside them.
- For $a < 0$ every sign above reverses, so take the minus outside first.

Always positive means $a > 0$ and $D < 0$, both. A question asking for a quadratic to be positive for all x is asking for exactly those two conditions, and forgetting the sign of a is what loses the mark.

89

IRRATIONAL INEQUALITIES

A-B9

A square root is defined only where its argument is non-negative, and it is never negative itself. Those two facts do all the work.

INEQUALITY	EQUIVALENT SYSTEM
$\sqrt{f} < g$	$f \geq 0, g > 0$ and $f < g^2$
$\sqrt{f} > g$	$f \geq 0$ with $g < 0$, or $f > g^2$ with $g \geq 0$
$\sqrt{f} < \sqrt{g}$	$f \geq 0$ and $f < g$
$\sqrt{f} > \sqrt{g}$	$g \geq 0$ and $f > g$

Squaring is legal only when both sides are non-negative. The second row splits into two cases precisely because a negative right side makes the inequality automatic, and missing that branch loses half the answer.

90

MODULUS & STEP FUNCTIONS

B-10-16

810

THE MODULUS

B-10

KEY

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

- $|x| \geq 0$ always, and $|x| = 0$ only at $x = 0$.
- $|x|^2 = x^2$ and $\sqrt{x^2} = |x|$, not x .
- $|xy| = |x||y|$, and $|\frac{x}{y}| = \frac{|x|}{|y|}$ for $y \neq 0$.

$\sqrt{x^2} = |x|$ is the identity that gets forgotten. Writing $\sqrt{x^2} = x$ is right only for $x \geq 0$, and the negative branch is exactly where the question is set.

91

MODULUS EQUATIONS

B-11

- $|x| = a$: two solutions $x = \pm a$ when $a > 0$, one when $a = 0$, none when $a < 0$.
- $|f(x)| = a$ with $a > 0$: solve $f = a$ and $f = -a$ and keep every root.
- $|f(x)| = g(x)$: solve $f = \pm g$ and then keep only the roots with $g(x) \geq 0$.
- $|f(x)| = |g(x)|$: solve $f = g$ and $f = -g$; no condition is needed.

The right side decides how much checking is needed. Against a constant or another modulus, nothing can go wrong. Against a general $g(x)$, every root has to be tested for $g \geq 0$, because a modulus is never negative.

92

MODULUS INEQUALITIES

B-12

KEY

$$|x| < a \Leftrightarrow -a < x < a, \quad |x| > a \Leftrightarrow x < -a \text{ or } x > a$$

- Both hold for every real a : for $a \leq 0$ the first is empty on both sides, and for $a < 0$ the second is the whole line on both sides. The useful case is $a > 0$, drawn above.
- $|f(x)| < g(x) \Leftrightarrow -g < f < g$, which already forces $g > 0$.
- $|f(x)| > g(x) \Leftrightarrow f > g \text{ or } f < -g$, and this holds for every g .

Less than gives one interval, greater than gives two, but only against a constant. For $|x| < a$ and $|x| > a$ that is a genuine check. Against a variable right side the shape is decided by f and g instead: $|x^2 - 5| < 1$ gives two separated intervals, and $|x| > x^2 - 2$ gives the single interval $(-2, 2)$.

93

TWO OR MORE MODULI

B-13

1. Find the critical points, where each expression inside a modulus is zero.
- ii. Those points cut the line into intervals; on each one every modulus opens with a fixed sign.
- iii. Solve the resulting plain inequality on each interval separately.
- iv. Keep only the part of each solution that lies inside the interval it came from, then take the union.

$|x-1| + |x-3| < 6$ has critical points 1 and 3, so three cases: $x < 1$, $1 \leq x < 3$, $x \geq 3$.

Intersecting with the interval is not optional. Each case is solved under an assumption about where x lies, so a root outside that interval is not a solution of the original problem, however correct the algebra was.

94

THE TRIANGLE INEQUALITY

B-14

KEY

$$||x| - |y|| \leq |x \pm y| \leq |x| + |y|$$

- $|x + y| = |x| + |y|$ exactly when x and y have the same sign or one is zero.
- $|x - y|$ is the distance between x and y on the line.
- The same inequality holds for complex numbers and for vectors, with equality when one is a non-negative real multiple of the other, which is the vector version of having the same sign.

Read it as a distance statement. $|x - a| < \delta$ says x is within δ of a , which turns most modulus questions into a picture on the number line and removes the need for cases altogether.

95

GREATEST INTEGER AND FRACTIONAL PART

B-15

- $[x]$ is the greatest integer not exceeding x ; $\{x\} = x - [x]$ lies in $[0, 1)$.
- $[x] \leq x < [x] + 1$ and $x - 1 < [x] \leq x$.
- $[x + n] = [x] + n$ for integer n , but $[x + y] \neq [x] + [y]$ in general.
- $[x] + [-x] = 0$ when x is an integer and -1 otherwise.

$[-2.3]$ is -3 , not -2 . The greatest integer goes down, not towards zero, and the negative case is the one the paper sets. Test any identity you half remember at $x = -2.3$ before using it.

96

SIGNUM, AND THE FOUR TOGETHER

B-16

KEY

$$\text{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 = \frac{|x|}{x} \ (x \neq 0) \\ -1, & x < 0 \end{cases}$$

FUNCTION	RANGE AND BEHAVIOUR
$ x $	$[0, \infty)$, continuous, corner at 0
$[x]$	$\mathbb{Z}$, a staircase, jumps at each integer
$\{x\}$	$[0, 1)$, a sawtooth of period 1
$\text{sgn}(x)$	$\{-1, 0, 1\}$, one jump at 0

All four are built from the same idea. Each splits the line at a set of points and behaves simply in between, which is why every question about them is answered by finding those points and working interval by interval.

97

LOGARITHMS

B-17-25

17

THE LOGARITHM AND WHEN IT EXISTS

C-17

KEY

$$\log_a N = x \Leftrightarrow a^x = N$$

with $a > 0$, $a \neq 1$ and $N > 0$. All three conditions are part of the definition.

- $\log_a 1 = 0$ and $\log_a a = 1$ for every legal base.
- $a^{\log_a N} = N$, the two operations undoing each other.
- log without a base means base 10; ln means base e .

Write the domain conditions down before solving. Every argument positive, every base positive and not 1. Solutions that break one of them are extraneous, and a logarithmic question almost always produces at least one.

98

THE LAWS

C-18

1. $\log_a(MN) = \log_a M + \log_a N$
- ii. $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$
- iii. $\log_a(M^p) = p \log_a M$
- iv. $\log_a(M^{\frac{1}{p}}) = \frac{1}{p} \log_a M$

The third law can change the domain. $\log_a x^2$ is defined for every $x \neq 0$, but $2 \log_a x$ only for $x > 0$. The correct identity is $\log_a x^2 = 2 \log_a |x|$, and the missing modulus is a standard trap.

99

CHANGE OF BASE AND WHAT FOLLOWS

C-19

KEY

$$\log_a N = \frac{\log_b N}{\log_b a}$$

- i. $\log_a b \cdot \log_b a = 1$, so the two are reciprocals.
- ii. $\log_b b \cdot \log_c c \cdot \log_d d = \log_a d$, a chain that telescopes.
- iii. $a^{\log_a c} = c^{\log_a a}$.
- iv. $\log_{a^p}(M^p) = \frac{p}{q} \log_a M$ needs $q \neq 0$.

Put everything on one base first. Almost every logarithmic equation becomes routine once all the logarithms share a base, and the base to choose is usually the one that appears most often, or the smallest.

99

THE GRAPH, AND WHY THE BASE MATTERS

C-20

The graph is the whole of section 22. Whether a logarithmic inequality keeps or reverses its direction is decided by which of these two pictures applies, so sketching the right one takes the place of memorising a rule.

21

LOGARITHMIC EQUATIONS

C-21

- i. State the domain: every argument positive, every base positive and not 1.
- ii. Bring all the logarithms to one base.
- iii. Combine them into a single logarithm using the laws.
- iv. Remove the logarithm by $\log_a M = \log_a N \Rightarrow M = N$, or by writing $a^x = N$.
- v. Reject every root that fails the domain.

$\log_3 x + \log_3(x-2) = 3$ needs $x > 2$; it gives $x(x-2) = 8$, so $x = 4$ or $x = -2$, and only $x = 4$ survives.

Combining logarithms enlarges the domain, which is why roots must be checked. An inequality in $\log_a(\dots)$ splits into $x > 1$ and $0 < x < 1$, each giving its own answer, and the union of the two is what the question wants.

22

LOGARITHMIC INEQUALITIES

C-22

KEY

$$\log_a M > \log_a N \Leftrightarrow \begin{cases} M > N, & a > 1 \\ M < N, & 0 < a < 1 \end{cases}$$

- i. Fix the domain first: every argument must be positive.
- ii. Split on the base: is $a > 1$ or is $0 < a < 1$?
- iii. Drop the logarithms, keeping or reversing the inequality accordingly.
- iv. Intersect the result with the domain.

If the base itself contains the variable, both cases must be solved. An inequality in $\log_a(\dots)$ splits into $x > 1$ and $0 < x < 1$, each giving its own answer, and the union of the two is what the question wants.

23

CHARACTERISTIC AND MANTISSA

C-23

Writing $\log_{10} N$ = characteristic + mantissa, the characteristic is $\lfloor \log_{10} N \rfloor$ and the mantissa is the fractional part, which lies in $[0, 1)$.

- For $N \geq 1$ the characteristic is one less than the number of digits before the decimal point.
- For $0 < N < 1$ it is negative, and is written in bar notation such as $\bar{3}.4771$.
- The mantissa depends only on the digits of N , not on where the decimal point sits.

The mantissa is always taken non-negative. That is why the extra root exists: $\log_{10} 0.003 = \bar{3} + 0.4771$, not -3.4771 , and treating the whole value as negative is what makes the digit count come out wrong.

24

COUNTING DIGITS AND ZEROS

C-24

- i. The number of digits in N is $\lfloor \log_{10} N \rfloor + 1$.
- ii. The number of zeros immediately after the decimal point in $N < 1$ is $\lfloor \log_{10} N \rfloor - 1$.
- iii. $\log_{10} 2 \approx 0.3010$, $\log_{10} 3 \approx 0.4771$, $\log_{10} 7 \approx 0.8451$.

2^{100} : $\log_{10} 2^{100} = 30.10$, so it has 31 digits and its leading digit is 1, since the mantissa 0.10 is below $\log_{10} 2$.

Three logarithms generate the rest. From $\log 2$, $\log 3$ and $\log 7$ you can build log of any integer up to 10 using $\log 5 = 1 - \log 2$, $\log 6 = \log 2 + \log 3$, $\log 8 = 3 \log 2$ and $\log 9 = 2 \log 3$.

25

EXPONENTIAL EQUATIONS AND INEQUALITIES

C-25

- $a^x = a^y \Leftrightarrow x = y$, for $a > 0$ and $a \neq 1$.
- $a^x > a^y \Leftrightarrow x > y$ when $a > 1$, and $x < y$ when $0 < a < 1$: the same split as for logarithms.
- $a^{f(x)} = b^{g(x)}$; take logarithms of both sides to any convenient base.
- An equation of the form $a^{2x} + ka^x + m = 0$ is a quadratic in $t = a^x$, with $t > 0$ compulsory.

The substitution $t = a^x$ carries a hidden condition. t must be positive, so a negative root of the quadratic is rejected before it is ever converted back. That rejection is usually where the marks are.

26

THE IDENTITIES

D-26

- i. $(a+b)^2 = a^2 + 2ab + b^2$ and $a^2 - b^2 = (a+b)(a-b)$
- ii. $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$
- iii. $a^3 \pm b^3 = (a \pm b)(a^2 \pm ab + b^2)$
- iv. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+bc+ca)$
- v. $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$
- vi. $a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2)$

27

WHEN THE SUM IS ZERO

D-27

KEY

$$a^3 + b^3 + c^3 = 3abc$$

Also $a^3 + b^3 + c^3 = -2(ab+bc+ca)$. And $(a^2 + b^2 + c^2)^2 = 2(a^4 + b^4 + c^4)$. The condition $a + b + c = 0$ is often hidden, for instance in $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$ with $xyz \neq 0$.

This one line closes a large family of questions. Whenever three quantities are given whose sum vanishes, check for it before doing anything else; the answer is usually one substitution away.

28

RATIOS AND PROPORTION

D-28

If $\frac{a}{b} = \frac{c}{d}$ then, provided no denominator vanishes:

- **Componendo:** $\frac{a+b}{b} = \frac{c+d}{d}$.
- **Dividendo:** $\frac{a-b}{b} = \frac{c-d}{d}$.
- **Componendo and dividendo:** $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.
- If $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \dots = k$ then each equals $\frac{\sum \lambda_1 a_1}{\sum \lambda_1 b_1}$ for any weights with non-zero denominator.

Componendo and dividendo is the fastest route through a surd equation. An equation of the form $\frac{\sqrt{P} + \sqrt{Q}}{\sqrt{P} - \sqrt{Q}} = k$ is solved in two lines by it, where squaring twice takes a page.

29

THE THREE MEANS

D-29

KEY

$$A = \frac{\sum a_i}{n}, \quad G = \left(\prod a_i\right)^{1/n}, \quad H = \frac{n}{\sum \frac{1}{a_i}}$$

- For two numbers, $A = \frac{a+b}{2}$, $G = \sqrt{ab}$, $H = \frac{2ab}{a+b}$.
- For two numbers $G^2 = AH$, so A, G, H are themselves in geometric progression.
- Given A and G , the two numbers are the roots of $x^2 - 2Ax + G^2 = 0$.

That quadratic is a second proof that $A \geq G$. The numbers were given as real, so its discriminant $4(A^2 - G^2)$ cannot be negative. The inequality follows from nothing more than the roots being real.

30

THE CHAIN $A \geq G \geq H$

D-30

KEY

$$\frac{a_1 + \dots + a_n}{n} \geq (a_1 \dots a_n)^{1/n} \geq \frac{n}{\frac{1}{a_1} + \dots + \frac{1}{a_n}}$$

The radius is A , the half chord is G , so $A > G$.

- $A \geq G$ needs the terms non-negative; the full chain needs them positive, since H divides by them.
- Equality throughout holds exactly when all the terms are equal.
- That equality case is what turns the inequality into a maximum or a minimum.

The equality case is the answer, not a footnote. A question asking for a least value is answered by applying the chain and then solving the equality condition, which pins down the point where the least value occurs.

31

STANDARD INEQUALITIES THAT FOLLOW

D-31

1. $x + \frac{1}{x} \geq 2$ for $x > 0$, and ≤ -2 for $x < 0$.
- ii. $(\sum a_i) \left(\sum \frac{1}{a_i} \right) \geq n^2$, so $(a+b+c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 9$.
- iii. $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}}$.
- iv. $\frac{a_1^m + \dots + a_n^m}{n} \geq \left(\frac{a_1 + \dots + a_n}{n} \right)^m$ for $m < 0$ or $m > 1$, reversed for $0 < m < 1$.

The first line is the one that appears every year. It is AM $\geq$ GM on the two numbers x and $1/x$, whose product is 1, and the same trick handles any pair whose product is a constant.

32

CAUCHY AND THE WEIGHTED MEAN

D-32

KEY

$$\left(\sum a_i b_i \right)^2 \leq \left(\sum a_i^2 \right) \left(\sum b_i^2 \right)$$

and for positive weights $m_1, \dots, m_n$,

$$\frac{\sum m_i a_i}{\sum m_i} \geq (a_1^{m_1} \dots a_n^{m_n})^{1/\sum m_i}$$

- Cauchy holds for all reals, not only positive ones.
- Equality in Cauchy needs the two sequences proportional.
- The weighted mean reduces to AM $\geq$ GM when all the weights are 1.

Advanced only. JEE Main asks for AM $\geq$ GM and the reciprocal inequality; these two appear in Advanced, usually as the last step of a longer argument rather than as the question itself.

33

SURDS AND RATIONALISING

D-33

- $\sqrt{a}\sqrt{b} = \sqrt{ab}$ holds for $a, b \geq 0$, and fails for two negatives.
- To rationalise $\frac{1}{\sqrt{a} + \sqrt{b}}$, multiply above and below by $\sqrt{a} - \sqrt{b}$.
- For a three-term surd, group two of them and rationalise twice.
- For a cube root, use $a^3 \pm b^3$ to choose the multiplier.

A surd equals a rational only trivially: if $a + b\sqrt{n} = c + d\sqrt{n}$ with a, b, c, d rational and $\sqrt{n}$ irrational, then

KEY

$$a = c \quad \text{and} \quad b = d$$

Comparing rational and irrational parts is a legitimate method. It is the surd version of comparing real and imaginary parts, and it turns one equation into two. It needs $\sqrt{n}$ genuinely irrational, which is why n must not be a perfect square.

34

POLYNOMIAL DIVISION

D-34

Dividing $P(x)$ by $D(x)$ gives a unique quotient and remainder with $\deg R < \deg D$:

KEY

$$P(x) = D(x)Q(x) + R(x)$$

- **Remainder theorem:** the remainder on division by $x-a$ is $P(a)$.
- **Factor theorem:** $x-a$ divides P exactly when $P(a) = 0$.
- Dividing by a quadratic leaves a remainder of the form $\alpha x + \beta$, so two values of x determine it.
- A polynomial of degree at most n with more than n roots is identically zero.

The last line settles identity questions in one step. If two polynomials of degree n agree at $n+1$ points they are the same polynomial, which is why checking a few values is a proof and not merely evidence.

35

WHICH METHOD THE INEQUALITY WANTS

E-35

WHAT YOU SEE	WHAT TO DO
a polynomial against zero	factorise, then wavy curve
a quotient against zero	one side, then wavy curve on P/Q
a quadratic, sign for all x </	