

CONCEPT MAP

MATRICES

5 TOPICS • 38 CORE IDEAS • ONE SHEET

A THE MATRIX ITSELF 01-08

01 WHAT A MATRIX IS, AND ITS ORDER A-01

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

column 2
row 2
order: 3 × 4; rows first, then columns

- A **matrix** is a rectangular array of numbers arranged in rows and columns. It is a bookkeeping device, not a number, so it has no value of its own.
- If it has m rows and n columns its **order** is $m \times n$, always rows first. The array then holds mn entries.
- The entry in row i and column j is written a_{ij} , and the whole matrix is written $A = [a_{ij}]_{m \times n}$.
- A matrix is often defined by a rule, and then you build it entry by entry from that rule.

$$A = [a_{ij}]_{2 \times 3} \text{ With } a_{ij} = i + 2j \text{ gives } \begin{bmatrix} 3 & 5 & 7 \\ 4 & 6 & 8 \end{bmatrix}$$

The number of matrices of order $m \times n$ with entries from a set of k elements is k^{mn} . If instead you are told the number of entries is N , the possible orders correspond to the factor pairs of N , so a prime N allows only $1 \times N$ and $N \times 1$.

02 TYPES BY SHAPE A-02

NAME	DESCRIPTION
row matrix	one row, order $1 \times n$
column matrix	one column, order $m \times 1$
square matrix	$m = n$; then n is called its order
rectangular	$m \neq n$
zero matrix O	every entry zero, any order
horizontal	$m < n$, wider than it is tall
vertical	$m > n$, taller than it is wide

- In a square matrix the entries $a_{11}, a_{22}, \dots, a_{nn}$ form the **leading diagonal**, also called the principal diagonal.
- Only square matrices have a determinant, a trace, an inverse or powers, so checking squareness is the first move in most questions.

03 TYPES BY ENTRIES A-03

NAME	CONDITION (SQUARE A)
diagonal	$a_{ij} = 0$ for all $i \neq j$
scalar	diagonal with all diagonal entries equal
identity I_n	scalar with every diagonal entry 1
upper triangular	$a_{ij} = 0$ for all $i > j$
lower triangular	$a_{ij} = 0$ for all $i < j$
triangular	upper or lower

- A diagonal matrix is written $\text{diag}(d_1, \dots, d_n)$, and a scalar matrix is kI .
- A matrix that is both upper and lower triangular is diagonal.
- The determinant of a triangular or diagonal matrix is the product of its diagonal entries.

Diagonal matrices behave almost like numbers. They commute with one another, they multiply and add entry by entry along the diagonal, and $\text{diag}(d_i)^n = \text{diag}(d_i^n)$. This is why so many high power questions are really a change of viewpoint into diagonal form.

04 EQUALITY OF MATRICES A-04

- $A = B$ requires two things: the same order, and $a_{ij} = b_{ij}$ for every i and j .
- So one matrix equation of order $m \times n$ is really mn scalar equations at once, which is how unknowns are extracted.
- Matrices of different orders are never equal, and are never even compared.

$$\begin{bmatrix} x+y & 2 \\ 5 & x-y \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 2 \end{bmatrix} \text{ gives } x+y=6 \text{ and } x-y=2, \text{ so } x=4, y=2.$$

05 ADDITION AND SCALAR MULTIPLICATION A-05

- $A + B$ is defined only when A and B have the **same order**, and then $(A + B)_{ij} = a_{ij} + b_{ij}$.
- kA multiplies every entry by k , so $kA = [ka_{ij}]$ and $-A = (-1)A$.
- Subtraction is $A - B = A + (-B)$, again only for matching orders.

$$[kA] = k[A] \text{ for a square } A \text{ of order } n$$

The determinant rule is the classic trap. A scalar multiplies every one of the n rows, so it comes out of the determinant n times, not once. Compare $\text{tr}(kA) = k \text{tr } A$, where it does come out just once.

B6 LAWS OF ADDITION A-06

- Closure and commutativity:** $A + B$ has the same order, and $A + B = B + A$.
- Associativity:** $(A + B) + C = A + (B + C)$.
- Additive identity:** $A + O = O + A = A$, where O has the order of A .
- Additive inverse:** $A + (-A) = O$.
- Distributive over scalars:** $k(A + B) = kA + kB$ and $(k + l)A = kA + lA$.
- Cancellation holds:** $A + C = B + C \implies A = B$.

Addition is completely well behaved. Every law you expect from numbers survives, including cancellation. Multiplication is where the surprises live, and topic B is about exactly those failures.

07 TRACE A-07

For a square matrix, the **trace** is the sum of the leading diagonal entries.

$$\text{tr } A = \sum_{i=1}^n a_{ii}$$

PROPERTY	CONDITION
$\text{tr}(A \pm B) = \text{tr } A \pm \text{tr } B$	same order
$\text{tr}(kA) = k \text{tr } A$	any k
$\text{tr}(A^T) = \text{tr } A$	always
$\text{tr}(AB) = \text{tr}(BA)$	both products defined
$\text{tr } I_n = n$	identity of order n

$\text{tr}(AB) = \text{tr}(BA)$ even though $AB \neq BA$. Both sides equal $\sum_i a_{ij}b_{ji}$. It follows that $\text{tr}(AB - BA) = 0$, so no pair of real or complex square matrices can satisfy $AB - BA = I$. Note also that the trace does not extend to three factors: $\text{tr}(ABC) = \text{tr}(BCA)$ is true, but $\text{tr}(ABC) \neq \text{tr}(ACB)$ is false in general.

08 COUNTING MATRICES A-08

HOW MANY, OF ORDER n	COUNT
all matrices, entries from k values	k^{n^2}
diagonal	k^n
upper triangular, and lower separately	$k^{n(n+1)/2}$ each
symmetric	$k^{n(n+1)/2}$
skew symmetric, over k values with 0	$k^{n(n-1)/2}$
scalar	k

- Each count is k raised to the number of entries you are free to choose, so the whole method is: decide which entries are forced, and count the rest.
- In a skew symmetric matrix the diagonal is forced to zero and the lower triangle is forced by the upper, leaving $\frac{n(n-1)}{2}$ free entries. For that count to be a clean power the value set must contain 0 and be closed under negation, and the diagonal, triangular and scalar counts likewise need 0 in the set, since they force entries to be zero.
- Upper and lower triangular are counted separately. The matrices that are upper or lower triangular number $2^{n(n+1)/2} - k^n$, because the diagonal ones lie in both families.
- For a matrix of order $m \times n$ with mn entries fixed, the possible orders are the factor pairs of mn .

Symmetric matrices of order 3 with entries from $\{0, 1\}$: 0 free entries, so $2^5 = 64$.

B MULTIPLICATION 09-17

09 THE MULTIPLICATION RULE B-09

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \end{bmatrix}$$

row of the first against column of the second
($m \times n$)($n \times p$) = ($m \times p$)

Each entry of the product is a row of the first matrix run against a column of the second.

$$(AB)_{ij} = \sum_k a_{ik}b_{kj}$$

- AB exists only when the number of columns of A equals the number of rows of B . Those two are said to be **conformable**.
- If A is $m \times n$ and B is $n \times p$ then AB is $m \times p$: the inner dimensions cancel and the outer ones survive.
- Entry (i, j) costs n multiplications, so a small product is quick but the arithmetic must be done in order.

10 WHEN PRODUCTS EXIST B-10

A	B	PRODUCTS
$m \times n$	$n \times p$	AB is $m \times p$
$m \times n$	$p \times q, n \neq p$	AB undefined
$1 \times n$	$n \times 1$	AB is $1 \times 1, BA$ is $n \times n$
$n \times n$	$n \times n$	both exist, both $n \times n$

- AB and BA can both exist yet have different orders, so they need not even be comparable.
- They have the same order only when A is $m \times n$ and B is $n \times m$ with $m = n$, that is when both are square of the same order.

A row times a column is a single number, a column times a row is a whole matrix. With X of order $n \times 1, X^T X$ is 1×1 and, for real entries, equals the sum of their squares, while XX^T is $n \times n$ and has determinant zero for $n \geq 2$.

11 MULTIPLICATION DOES NOT COMMUTE B-11

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$
$$BA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

same factors, different products

- In general $AB \neq BA$, so the order of the factors must be preserved at every step.
- Matrices that do satisfy $AB = BA$ are called **commuting**. Any matrix commutes with I , with O , with kI , and with its own powers and its inverse.
- A and B commute if and only if $(A + B)^2 = A^2 + 2AB + B^2$, and equally if and only if $(A + B)(A - B) = A^2 - B^2$.

$$(A + B)^2 = A^2 + AB + BA + B^2$$

Every binomial identity has to be re-derived. Expand by brute force, keeping AB and BA apart, and only then use any commuting fact you have actually been given.

12 ZERO DIVISORS AND FAILED CANCELLATION B-12

- $AB = O$ does not force $A = O$ or $B = O$.
- $AB = AC$ with $A \neq O$ does not force $B = C$, so cancellation is illegal.
- Both do become valid the moment A is invertible, since then you may multiply through by A^{-1} on the correct side.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = O \text{ with neither factor zero.}$$

What survives is the determinant. From $AB = O$ you get $|A||B| = 0$, so at least one of the two is singular. That is the correct deduction, and it is often all a question needs.

13 LAWS THAT DO HOLD B-13

- Associativity:** $(AB)C = A(BC)$ whenever the products are defined.
- Distributivity:** $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$.
- Identity:** $A_{m \times n}I_n = A$ and $I_m A_{m \times n} = A$.
- Zero:** $AO = O$ and $OA = O$, with the zero matrices of the right orders.
- Scalars slide:** $k(AB) = (kA)B = A(kB)$.
- Determinant:** $|AB| = |A||B|$ for square A and B of the same order.

Associativity is a computing tool. To evaluate ABX where X is a column, take $A(BX)$ and you multiply a matrix by a column twice instead of forming a full matrix product. The same idea turns $(AB)^n = A(BA)^{n-1}B$ into a short calculation when BA is simple.

14 POWERS OF A SQUARE MATRIX B-14

- $A^0 = A, A^{n+1} = A^n A$, and $A^0 = I$ by convention for a square A .
- $A^m A^n = A^{m+n}$ and $(A^m)^n = A^{mn}$, since a matrix always commutes with its own powers.
- $(AB)^n = A^n B^n$ is guaranteed only when A and B commute, so in general expect $(AB)^n \neq A^n B^n$.
- $|A^n| = |A|^n$, and $\text{tr}(A^n)$ has no such simple rule.

TYPE	POWERS
idempotent, $A^2 = A$	$A^n = A$ for all $n \geq 1$
involutory, $A^2 = I$	$A^n = I$ or A as n is even or odd
nilpotent, index k	$A^n = O$ for all $n \geq k$
periodic, period k	$A^{n+k} = A^n$ for $n \geq 1$, so powers cycle
$\text{diag}(d_i)$	$\text{diag}(d_i^n)$

15 MATRIX POLYNOMIALS B-15

- For a polynomial $p(x) = c_0 + c_1x + \dots + c_nx^n$, the matrix $p(A) = c_0I + c_1A + \dots + c_nA^n$ is defined for any square A . The constant term must be written as c_0I , never as c_0 alone.
- A is a **root** of p if $p(A) = O$. Every relation such as $A^2 = A + I$ is a statement of this kind, and it lets every higher power be reduced.
- Two polynomials in the same matrix always commute, since both are built from powers of A .

If $A^2 = A + I$ then $A^3 = A \cdot A^2 = A^2 + A = 2A + I$, and the pattern continues with Fibonacci coefficients.

A polynomial relation is the fastest route to an inverse. From $A^2 - 3A + 2I = O$ read $A(3I - A) = 2I$, so $A^{-1} = \frac{1}{2}(3I - A)$. This works whenever the constant term is a non zero multiple of I , and that constant term is exactly $\pm|A|$ for the characteristic polynomial.

16 POWERS OF STANDARD MATRICES B-16

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 \\ na & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$$

- If $A = I + N$ with N nilpotent and N commuting with I , the binomial theorem applies and terminates: $A^n = I + nN + \binom{n}{2}N^2 + \dots$, stopping where N dies.
- If $A^2 = kA$ then $A^n = k^{n-1}A$ for $n \geq 2$, and for $n = 1$ as well when $k \neq 0$, which covers most 3×3 power questions built from a column times a row.

Method for any A^n question. Compute A^2 and A^3 first. If a pattern is visible, prove it by induction. If not, look for $A^2 = kA$, for $A = I + N$, or reach for Cayley and Hamilton in section 32.

17 COMMUTING FAMILIES B-17

- The matrices that commute with a given A always include I , every scalar matrix kI , every power A^n , and A^{-1} when it exists. So every polynomial in A commutes with A .
- For a 2×2 matrix A that is not scalar, these are the only ones: everything commuting with A has the form $aI + bA$.
- A matrix commutes with **every** matrix of its order if and only if it is scalar.
- If A and B commute then so do A^n and B^n , and the binomial theorem holds for $(A + B)^n$.

$$X \text{ commutes with } \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ exactly when } X = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}, \text{ which is } aI + bA.$$

Read a commuting condition as a linear system. Write X with four unknown entries, form $AX - XA = O$, and equate entries. That is four scalar equations, and their solution set is the family you were asked for.

C TRANSPOSE & TYPES 18-27

18 TRANSPOSE C-18

The **transpose** A^T , also written A' , turns rows into columns: if A is $m \times n$ then A^T is $n \times m$ with $(A^T)_{ij} = a_{ji}$.

- $(A^T)^T = A$.
- $(A + B)^T = A^T + B^T$, for matching orders.
- $(kA)^T = kA^T$.
- $(AB)^T = B^T A^T$, and the order reverses.
- $(A_1 A_2 \dots A_k)^T = A_k^T \dots A_2^T A_1^T$.
- $|A^T| = |A|$ and $\text{tr}(A^T) = \text{tr } A$ for square A .
- $(A^T)^{-1} = (A^{-1})^T$ when A is invertible.

The reversal rule is the engine of this whole topic. Almost every proof about a special type begins with taking the transpose of both sides and applying it, so it is worth being automatic.

19 SYMMETRIC MATRICES C-19

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \text{ symmetric: } A^T = A, \text{ mirror entries equal.}$$
$$\begin{bmatrix} a & b \\ -b & c \end{bmatrix} \text{ skew: } A^T = -A, \text{ off-diagonal all zero.}$$

- A is **symmetric** if $A^T = A$, that is $a_{ij} = a_{ji}$. Only square matrices qualify.
- The entries are mirrored across the leading diagonal, and the diagonal itself is unrestricted.
- A symmetric matrix of order n has $\frac{n(n+1)}{2}$ independent entries.
- AA^T and $A^T A$ are symmetric for **every** matrix A , square or not, and $A + A^T$ is symmetric for every square A .
- If A is symmetric then so are kA , A^n and A^{-1} .

The sum of two symmetric matrices is symmetric, but the product need not be. For symmetric A and B , $(AB)^T = B^T A^T = BA$, so AB is symmetric exactly when A and B commute.

20 SKEW SYMMETRIC MATRICES C-20

- A is **skew symmetric** if $A^T = -A$, that is $a_{ij} = -a_{ji}$.
- Putting $i = j$ gives $a_{ii} = -a_{ii}$, so **every diagonal entry is zero**.
- The number of independent entries is $\frac{n(n-1)}{2}$.
- $A - A^T$ is skew symmetric for every square A .
- If A is skew symmetric then A^n is skew symmetric for odd n and symmetric for even n .

$$\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix} \text{ is skew symmetric of order 3, with 3 free entries.}$$

The odd order result. If A is skew symmetric of odd order n then $|A| = |A^T| = |-A| = (-1)^n |A|$, so $|A| = -|A|$, so $|A| = 0$ and A is singular. For even order $|A|$ is a perfect square, and in particular it can be non zero.

21 THE SYMMETRIC PLUS SKEW DECOMPOSITION C-21

Every square matrix splits in exactly one way into a symmetric part and a skew symmetric part.

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

symmetric skew symmetric

- Existence is the identity above; uniqueness follows by transposing $A = P + Q$ and solving the pair.
- The two parts are called $P = \frac{1}{2}(A + A^T)$ and $Q = \frac{1}{2}(A - A^T)$, and $\text{tr } Q = 0$ always.
- Since $\text{tr } A = \text{tr } P$, the whole trace sits in the symmetric part.
- The decomposition needs division by 2, so it is stated for real or complex entries.

This is a standard three mark question. Write A^T , add and halve for P , subtract and halve for Q , then verify $P + Q = A$ before writing the answer. The check costs one line and catches every sign slip.

22 SANDWICH RESULTS C-22

Expressions of the form $B^T A B$ inherit the type of the middle matrix, and this single pattern answers a long list of proof questions.

- A symmetric $\implies B^T A B$ symmetric, for any B of a conformable order.
- A skew symmetric $\implies B^T A B$ skew symmetric.
- $B^T B$ and $B B^T$ are symmetric for every B .
- A Hermitian $\implies B^H A B$ Hermitian, with the same argument with θ in place of T .

$$(B^T A B)^T = B^T A^T (B^T)^T = B^T A^T B$$

One line proves all four. Transpose the sandwich by the reversal rule, and the outer factors come back in their original places, so only A^T changes. Whatever A^T equals, the sandwich equals the same multiple of itself. Note that B need not be square: if B is $n \times p$ and A is $n \times n$, the sandwich is $p \times p$.

23 ORTHOGONAL MATRICES C-23

- A square matrix is **orthogonal** if $AA^T = A^T A = I$, equivalently $A^{-1} = A^T$.
- Then $|A|^2 = 1$, so $|A| = \pm 1$ and A is always invertible.
- The rows form a set of mutually perpendicular unit vectors, and so do the columns.
- Products of orthogonal matrices are orthogonal, and so are their transposes and inverses.

The rotation matrix $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ is orthogonal with determinant 1.

For a 2×2 matrix, $AA^T = I$ alone is enough, and in general each of $AA^T = I$ and $A^T A = I$ implies the other for square matrices. The two sided statement is kept because it is the one that generalises.

24 IDEMPOTENT, INVOLUTORY, NILPOTENT, PERIODIC C-24

TYPE	DEFINITION	CONSEQUENCE
idempotent	$A^2 = A$	$ A = 0$ or 1; $I - A$ is idempotent too
involutory	$A^2 = I$	$ A = \pm 1$; $A^{-1} = A$
nilpotent	$A^k = O$ for some k	$ A = 0$; k is the index
periodic	$A^{k+1} = A$ for some k	k is the period
singular	$ A = 0$	no inverse exists
non singular	$ A \neq 0$	A^{-1} exists and is unique