

[IIT-JEE MATHEMATICS]

OLYMPIAD QUALIFIER

2026

IOQM *Solutions*

MATHEMATICS

ALL THIRTY

WORKED IN FULL

IN THIS BOOK

QUESTIONS	30	EACH WITH ITS SOLUTION
MAXIMUM MARKS	100	
ANSWER KEY	AT THE END OF THE BOOK	

[HOW TO READ]

Working, not commentary

Every question of the paper is printed again here with its solution directly below it, so nothing has to be looked up in a second document while you read.

The solutions are written as working. A line of algebra that speaks for itself is left to speak for itself, and a sentence appears only where a sentence is the shortest way to say the thing. The tools are the ordinary school ones: the sine and cosine rules, the circle theorems, the factor theorem, arithmetic of remainders, and counting by cases.

Every geometry question carries a figure drawn to the data of that question, and the two questions where a graph settles the matter carry a graph.

The answer to each question stands on a gold plate at the foot of its working, and all thirty are collected in the key at the end.

[SOLUTIONS]

Thirty questions, worked *in full*

Section A

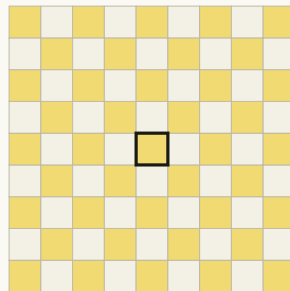
QUESTIONS 1 TO 10 · 2 MARKS EACH

Q 01 Combinatorics · CHESSBOARD COLOURING

2 MARKS

The central square of a 9×9 chessboard is white. How many white squares are there on the board? (The squares of the chessboard are coloured alternately black and white.)

SOLUTION



The 41 gold cells are the white ones; the centre is outlined.

Number the rows and columns 1 to 9 and give the square (i, j) the colour decided by the parity of $i + j$. The centre is $(5, 5)$ with $i + j = 10$ even, so

$$(i, j) \text{ white} \iff i + j \text{ even} \iff i, j \text{ both odd or both even.}$$

$$\#\{\text{odd}\} = |\{1, 3, 5, 7, 9\}| = 5, \quad \#\{\text{even}\} = |\{2, 4, 6, 8\}| = 4$$

$$\text{white} = 5 \times 5 + 4 \times 4 = 25 + 16 = 41$$

ANSWER 41

Q 02 Number Theory · DIVISIBILITY AND REMAINDERS

2 MARKS

An integer M is divisible by 4 but not by 256. What is the number of distinct possible remainders when M is divided by 256?

SOLUTION

Write $M = 256q + r$ with $0 \leq r < 256$. Since $4 \mid M$ and $4 \mid 256$,

$$r = M - 256q \implies 4 \mid r \implies r \in \{0, 4, 8, \dots, 252\}$$

$$\#\{0, 4, \dots, 252\} = \frac{252 - 0}{4} + 1 = 64$$

Also $r = 0 \iff 256 \mid M$, which is excluded, so $r \neq 0$. Every other value is reached: for $r \in \{4, 8, \dots, 252\}$ the integer $M = r$ itself is divisible by 4 and not by 256.

$$64 - 1 = 63$$

ANSWER 63

Q 03 Algebra · INTEGER OPTIMISATION

2 MARKS

If $x_1, x_2, \dots, x_{49}$ are nonzero integers such that

$$\sum_{i=1}^{49} x_i = 0,$$

then what is the minimum possible value of $\sum_{i=1}^{49} x_i^2$?

SOLUTION

Each x_i is a nonzero integer, so $x_i^2 \geq 1$ and

$$\sum x_i^2 \geq 49$$

For every integer, $x^2 - x = x(x - 1)$ is even, so $x^2 \equiv x \pmod{2}$. Hence

$$\sum x_i^2 \equiv \sum x_i = 0 \pmod{2} \implies \sum x_i^2 \text{ is even} \implies \sum x_i^2 \geq 50$$

If $\sum x_i^2 = 50$ then 48 of the squares equal 1 and one equals 2, and $x^2 = 2$ has no integer solution. If $\sum x_i^2 = 51$, the sum is odd. So the next possible value is

$$48 \times 1 + 4 = 52$$

It is attained by

$$x = +1 \text{ (23 times), } \quad x = -1 \text{ (25 times), } \quad x = +2 \text{ (once)}$$

$$\sum x_i = 23 - 25 + 2 = 0, \quad \sum x_i^2 = 23 + 25 + 4 = 52$$

ANSWER 52

Q 04 Combinatorics · DIGIT ARRANGEMENT

2 MARKS

All six digits of three 2-digit numbers are different. If N is the largest possible sum of three such numbers, what is the sum of the digits of N ?

SOLUTION

Let the three numbers be $\overline{a_1b_1}$, $\overline{a_2b_2}$, $\overline{a_3b_3}$ with all six digits distinct. Then

$$N = 10(a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)$$

A tens digit is worth ten times a units digit, so the three largest digits go to the tens places and the next three to the units places:

$$\{a_1, a_2, a_3\} = \{9, 8, 7\}, \quad \{b_1, b_2, b_3\} = \{6, 5, 4\}$$

$$N = 10(9 + 8 + 7) + (6 + 5 + 4) = 240 + 15 = 255$$

$$2 + 5 + 5 = 12$$

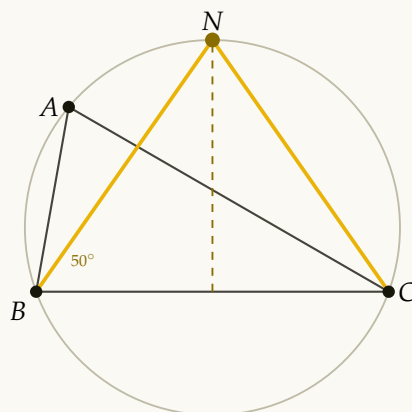
ANSWER 12

Q 05 Geometry · CIRCLE AND ARC ANGLES

2 MARKS

In triangle ABC , we are given that $\angle CAB = 80^\circ$. Let the perpendicular bisector of BC meet the circumcircle of triangle ABC at N , where A and N lie on the same side of the chord BC . What is the measure of $\angle NBC$ in degrees?

SOLUTION



N is the midpoint of arc BAC , so arc $NC = 100^\circ$.

The perpendicular bisector of a chord passes through the centre, so it meets the circle at the two midpoints of the arcs BC . As N is on the same side as A , N is the midpoint of the arc BC that contains A .

$$\text{arc } BC \text{ (not containing } A) = 2\angle CAB = 160^\circ$$

$$\text{arc } BAC = 360^\circ - 160^\circ = 200^\circ$$

$$\text{arc } BN = \text{arc } NC = \frac{1}{2}(200^\circ) = 100^\circ$$

$$\angle NBC = \frac{1}{2} \text{arc } NC = \frac{1}{2}(100^\circ) = 50^\circ$$

ANSWER 50

Q 06 Number Theory · DIGIT EQUATION

2 MARKS

Find the number of 2-digit positive integers n such that

$$n = 26 + (a \times b),$$

where a and b are the two digits of n .

SOLUTION

$$n = 10a + b, \quad 10a + b = 26 + ab$$

$$10a - ab = 26 - b \implies b(a - 1) = 10a - 26$$

$$b = \frac{10a - 26}{a - 1} = \frac{10(a - 1) - 16}{a - 1} = 10 - \frac{16}{a - 1}$$

So $(a - 1) \mid 16$ with $2 \leq a \leq 9$:

$$a - 1 \in \{1, 2, 4, 8\} \implies a \in \{2, 3, 5, 9\}, \quad b \in \{-6, 2, 6, 8\}$$

A digit satisfies $0 \leq b \leq 9$, so $a = 2$ is rejected and

$$n = 32, \quad n = 56, \quad n = 98$$

$$32 = 26 + 3 \cdot 2, \quad 56 = 26 + 5 \cdot 6, \quad 98 = 26 + 9 \cdot 8$$

ANSWER 3

Q 07 Combinatorics · RESTRICTED DIGIT STRINGS

2 MARKS

Find the number of positive integers n satisfying all the following conditions:

- (a) The digits of n lie in the set $\{1, 2, 4, 8\}$. (Digits may be repeated.)
- (b) The sum of the digits is 14.
- (c) If 1 occurs as a digit, then it can occur only immediately to the right of 8.

SOLUTION

Reading the number as blocks. Condition (c) says a digit 1 must have an 8 just before it. So tie every 1 to the 8 on its left. An 8 can carry at most one 1, because in 811 the second 1 has a 1 before it, not an 8. Every digit of n therefore belongs to exactly one block from

$$2, \quad 4, \quad 8, \quad 81$$

and reading n from left to right is the same as reading an ordered list of these blocks. Their digit sums are

$$2, \quad 4, \quad 8, \quad 9$$

and the list must total 14.

The digit 1 cannot appear at all. Only the block 81 has an odd sum. An odd number of them would make the total odd, but the total is 14, which is even. So their number is even, and two of them already give $9 + 9 = 18 > 14$. Hence there are none:

$$\text{no block } 81 \implies \text{the blocks are only } 2, 4, 8$$

What is left to count. Ordered lists of 2, 4, 8 with total 14. Halving every entry, these are exactly the ordered lists of 1, 2, 4 with total 7. Let $g(m)$ be the number of such lists totalling m . Splitting on the first entry,

$$g(m) = g(m-1) + g(m-2) + g(m-4), \quad g(0) = 1, \quad g(m) = 0 \text{ for } m < 0$$

$$g(1) = g(0) = 1$$

$$g(2) = g(1) + g(0) = 1 + 1 = 2$$

$$g(3) = g(2) + g(1) = 2 + 1 = 3$$

$$g(4) = g(3) + g(2) + g(0) = 3 + 2 + 1 = 6$$

$$g(5) = g(4) + g(3) + g(1) = 6 + 3 + 1 = 10$$

$$g(6) = g(5) + g(4) + g(2) = 10 + 6 + 2 = 18$$

$$g(7) = g(6) + g(5) + g(3) = 18 + 10 + 3 = 31$$

NOTE The shortest of the 31 numbers have three digits, namely the six arrangements of 2, 4 and 8, and the longest is 2222222.

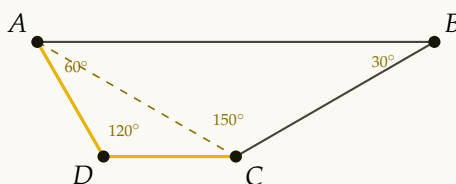
ANSWER 31

Q 08 Geometry · TRAPEZIUM AND ANGLES

2 MARKS

In trapezium $ABCD$, it is given that $AB \parallel CD$. Assume that $AB = 3CD$, $CD = DA$, and $\angle CDA = 120^\circ$. If the largest angle of $ABCD$ is x° and the smallest angle is y° , what is the value of x/y ?

SOLUTION



$AB = 3$, $CD = DA = 1$, and the diagonal $AC = \sqrt{3}$ makes ABC isosceles.

Take $CD = DA = 1$, so $AB = 3$. From $AB \parallel CD$,

$$\angle DAB = 180^\circ - \angle CDA = 60^\circ$$

In triangle ACD , $DA = DC = 1$ and $\angle ADC = 120^\circ$, so

$$\angle DAC = \angle DCA = 30^\circ, \quad AC^2 = 1 + 1 - 2 \cos 120^\circ = 3$$

$$\angle CAB = \angle DAB - \angle DAC = 60^\circ - 30^\circ = 30^\circ$$

In triangle ABC with $AB = 3$, $AC = \sqrt{3}$ and included angle 30° ,

$$BC^2 = 9 + 3 - 2(3)(\sqrt{3}) \cos 30^\circ = 12 - 9 = 3 \implies BC = \sqrt{3} = AC$$

$$\angle ABC = \angle BAC = 30^\circ, \quad \angle BCA = 120^\circ$$

$$\angle BCD = \angle BCA + \angle ACD = 120^\circ + 30^\circ = 150^\circ$$

$$\text{angles} = 60^\circ, 30^\circ, 150^\circ, 120^\circ; \quad \frac{x}{y} = \frac{150}{30} = 5$$

ANSWER 5

Q 09 Number Theory · DIGIT SUM EXTREMUM

2 MARKS

Let N be the smallest positive integer whose digits add up to 2026. What is the leading digit of $N + 1$?

SOLUTION

A number with k digits has digit sum at most $9k$, so

$$9k \geq 2026 \implies k \geq \frac{2026}{9} = 225.11 \dots \implies k \geq 226$$

Among 226-digit numbers the smallest is obtained by making the leading digit as small as possible and filling the rest with 9:

$$2026 = 9 \times 225 + 1 \implies N = 1 \underbrace{99 \dots 9}_{225}$$

$$N + 1 = 2 \underbrace{00 \dots 0}_{225}$$

ANSWER 2

Q 10 Number Theory · SUMS OF TWO SQUARES

2 MARKS

What is the number of integers in the set $\{0, \dots, 20\}$ which can be expressed as the sum of two square integers?

SOLUTION

Only $0^2, 1^2, 2^2, 3^2, 4^2$ can appear, that is 0, 1, 4, 9, 16. All sums of two of them that are at most 20:

+	0	1	4	9	16
0	0	1	4	9	16
1	1	2	5	10	17
4	4	5	8	13	20
9	9	10	13	18	—
16	16	17	20	—	—

$\{0, 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20\}$

$$|\cdot| = 13$$

ANSWER 13

Section B QUESTIONS 11 TO 20 · 3 MARKS EACH

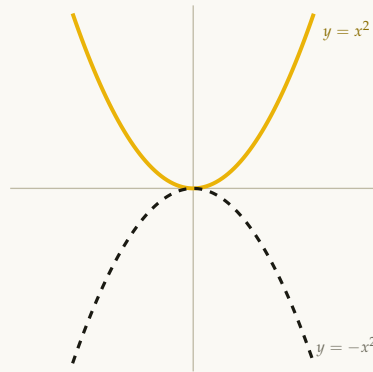
Q 11 Algebra · POLYNOMIAL FUNCTIONAL EQUATION

3 MARKS

Find the number of non-constant polynomials $P(x)$, with real coefficients, such that

$$P(x^2) = P(P(x)).$$

SOLUTION



The only two solutions, reflections of each other in the x -axis.

Let $\deg P = n \geq 1$. Comparing degrees,

$$\deg P(x^2) = 2n, \quad \deg P(P(x)) = n^2 \implies n^2 = 2n \implies n = 2$$

Put $P(x) = ax^2 + bx + c$ with $a \neq 0$:

$$P(x^2) = ax^4 + bx^2 + c$$

$$P(P(x)) = a(ax^2 + bx + c)^2 + b(ax^2 + bx + c) + c$$

Comparing coefficients,

$$x^4: \quad a^3 = a \implies a^2 = 1$$

$$x^3: \quad 2a^2b = 0 \implies b = 0$$

$$x^2: \quad a(2ac + b^2) + ab = b \implies 2a^2c = 0 \implies c = 0$$

$$P(x) = x^2 \quad \text{or} \quad P(x) = -x^2$$

Both work: $(-x^2)$ gives $P(x^2) = -x^4$ and $P(P(x)) = -(-x^2)^2 = -x^4$.

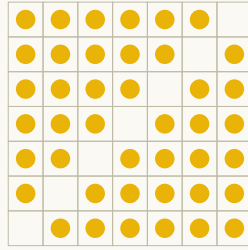
ANSWER 2

Q 12 Combinatorics · PARITY ON A GRID

3 MARKS

A 7×7 board is divided into 49 unit squares. We place checkers on the board, at most one per square. Find the largest number of checkers that can be placed on the unit squares so that each row, as well as each column, contains an even number of checkers.

SOLUTION



The diagonal left empty; every row and column then carries 6 checkers.

Each row holds an even number out of 7 squares, so each row holds at most 6:

$$\text{total} \leq 7 \times 6 = 42$$

Suppose the total is 42. Then every row holds exactly 6, so every row has exactly one empty square, giving 7 empty squares in all. If column j has e_j empty squares, its checker count is $7 - e_j$, which must be even, so

$$e_j \text{ is odd} \implies e_j \geq 1 \implies \sum_{j=1}^7 e_j \geq 7$$

and the sum is exactly 7, forcing $e_j = 1$ for every j . So the empty squares form one per row and one per column, for instance the main diagonal, and then every row and every column holds 6 checkers.

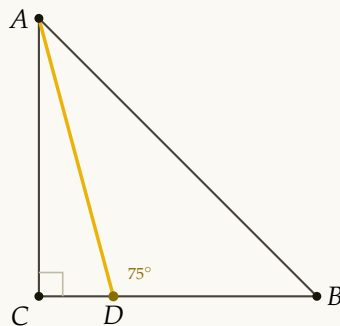
ANSWER 42

Q 13 Geometry · RIGHT ISOSCELES TRIANGLE

3 MARKS

In an isosceles triangle ABC , with $\angle ACB = 90^\circ$, the point D is on the side BC such that $\angle ADC = 75^\circ$. If the area of triangle ADC is 81, what is the length of segment BD ?

SOLUTION



$$CD = a(2 - \sqrt{3}) \text{ and } BD = a(\sqrt{3} - 1).$$

Let $CA = CB = a$. Triangle ACD is right angled at C , so

$$\tan 75^\circ = \frac{AC}{CD} \implies CD = \frac{a}{\tan 75^\circ}$$

$$\tan 75^\circ = 2 + \sqrt{3} \implies \frac{1}{\tan 75^\circ} = \frac{1}{2 + \sqrt{3}} = 2 - \sqrt{3}$$

$$CD = a(2 - \sqrt{3})$$

$$[ADC] = \frac{1}{2} AC \cdot CD = \frac{1}{2} a^2 (2 - \sqrt{3}) = 81 \implies a^2 = \frac{162}{2 - \sqrt{3}} = 162(2 + \sqrt{3})$$

$$BD = BC - CD = a - a(2 - \sqrt{3}) = a(\sqrt{3} - 1)$$

$$BD^2 = a^2(\sqrt{3} - 1)^2 = 162(2 + \sqrt{3})(4 - 2\sqrt{3}) = 324(2 + \sqrt{3})(2 - \sqrt{3}) = 324$$

$$BD = 18$$

ANSWER 18

Q 14 Number Theory · DIGIT REVERSAL AND FACTORS

3 MARKS

Let A be a 3-digit number with distinct nonzero digits and let B be the number obtained by reversing the digits of A . Determine the largest possible prime factor of $|A - B|$.

SOLUTION

$$A = 100a + 10b + c, \quad B = 100c + 10b + a$$

$$A - B = 99(a - c) \implies |A - B| = 99|a - c| = 3^2 \times 11 \times |a - c|$$

The digits are distinct and nonzero, so $a \neq c$ and $1 \leq |a - c| \leq 8$. The prime factors of $|A - B|$ are 3, 11, and the prime factors of $|a - c|$, which are at most 7. Hence the largest prime factor is 11, and it occurs for every choice, for example

$$A = 213, B = 312, |A - B| = 99 = 3^2 \times 11$$

ANSWER 11

Q 15 Number Theory · DIOPHANTINE OPTIMISATION

3 MARKS

Let a, b, c, d be positive integers such that

$$a^2 + b^2 - cd^2 = 2026.$$

Find the minimum possible value of $a + b + c + d$.

SOLUTION

$$a^2 + b^2 = 2026 + cd^2 \geq 2027$$

Write $s = a + b$ and $T = a + b + c + d$. Since $a, b \geq 1$,

$$a^2 + b^2 \leq (s - 1)^2 + 1$$

$$(a + b)^2 \geq a^2 + b^2 \implies s \geq \sqrt{2027} > 45 \implies s \geq 46$$

Also $c + d \geq 2$, so $T \geq 48$. Test the small values, using $ab = \frac{s^2 - (a^2 + b^2)}{2}$ where needed.

$$T = 48 : s = 46, c = d = 1, a^2 + b^2 = 2027 > 45^2 + 1 = 2026 \quad \text{impossible}$$

$$T = 49 : (s, c + d) = (47, 2) \Rightarrow a^2 + b^2 = 2027, ab = \frac{2209 - 2027}{2} = 91$$

$$\text{disc} = 47^2 - 4(91) = 1845, \quad 42^2 = 1764 < 1845 < 1849 = 43^2 \quad \text{impossible}$$

$$(s, c + d) = (46, 3) \Rightarrow a^2 + b^2 \in \{2028, 2030\} > 2026 \quad \text{impossible}$$

$$T = 50 : (48, 2) : ab = \frac{2304 - 2027}{2} \notin \mathbb{Z}; \quad (47, 3) : ab = \frac{2209 - 2028}{2} \notin \mathbb{Z}, \frac{2209 - 2030}{2} \notin \mathbb{Z}$$

$$(46, 4) : a^2 + b^2 \in \{2029, 2034, 2035\} > 2026 \quad \text{impossible}$$

$$T = 51 : a = 2, b = 45, c = 3, d = 1 : 4 + 2025 - 3 = 2026$$

ANSWER 51

Q 16 Algebra · RECURRENCE SEQUENCE

3 MARKS

A sequence $a_1, a_2, a_3, \dots$ of real numbers satisfies

$$\frac{a_{n+3} - a_{n+2}}{a_n - a_{n+1}} = \frac{a_{n+3} + a_{n+2}}{a_n + a_{n+1}}$$

for all $n \geq 1$. Suppose $a_{55} = 6$, $a_{66} = 2$, and $a_{77} = 1$. Let N denote the sum

$$N = a_1^2 + a_2^2 + \dots + a_{2026}^2.$$

What is the sum of the digits of N ?

SOLUTION

Cross multiplying,

$$(a_{n+3} - a_{n+2})(a_n + a_{n+1}) = (a_{n+3} + a_{n+2})(a_n - a_{n+1})$$

$$a_{n+3}a_{n+1} - a_{n+2}a_n = -a_{n+3}a_{n+1} + a_{n+2}a_n$$

$$2a_{n+3}a_{n+1} = 2a_{n+2}a_n \implies a_{n+3}a_{n+1} = a_{n+2}a_n$$

Replacing n by $n + 1$ in the last relation,

$$a_{n+4}a_{n+2} = a_{n+3}a_{n+1} = a_{n+2}a_n \implies a_{n+4} = a_n$$

so the sequence has period 4.

$$77 = 4(19) + 1, \quad 66 = 4(16) + 2, \quad 55 = 4(13) + 3$$

$$a_1 = a_{77} = 1, \quad a_2 = a_{66} = 2, \quad a_3 = a_{55} = 6$$

$$a_4a_2 = a_3a_1 \implies 2a_4 = 6 \implies a_4 = 3$$

$$1, 2, 6, 3, 1, 2, 6, 3, \dots$$

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1 + 4 + 36 + 9 = 50$$

$$2026 = 4(506) + 2$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 6 & 3 \\ \hline \end{array} \times 506 \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}$$

one block, $1 + 4 + 36 + 9 = 50$ left over, $1 + 4 = 5$

$2026 = 4 \times 506 + 2$, so the first 2024 terms make 506 whole blocks and only a_1 and a_2 are left at the end.

$$N = 506(50) + 1^2 + 2^2 = 25300 + 5 = 25305$$

$$2 + 5 + 3 + 0 + 5 = 15$$

ANSWER 15

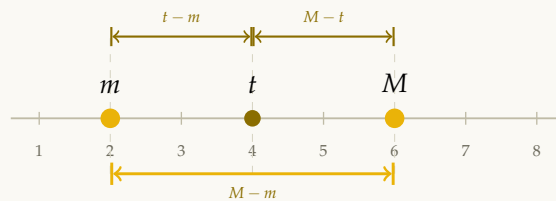
Q 17 **Combinatorics** · ORDERED TRIPLES

3 MARKS

Find the number of ordered triples (x, y, z) of positive integers such that $1 \leq x, y, z \leq 8$ and

$$|x - y| + |y - z| + |z - x| = 8.$$

SOLUTION



The two short jumps together cover the gap once and the long jump covers it again, so the three distances always add to $2(M - m)$ and the middle value t never matters.

Let $M = \max(x, y, z)$ and $m = \min(x, y, z)$, and let t be the third value, so $m \leq t \leq M$. Then

$$|x - y| + |y - z| + |z - x| = (M - t) + (t - m) + (M - m) = 2(M - m)$$

$$2(M - m) = 8 \implies M - m = 4$$

With $1 \leq m < M \leq 8$ and $M = m + 4$,

$$m \in \{1, 2, 3, 4\} \quad (4 \text{ windows})$$

For a fixed window $\{m, m + 1, \dots, m + 4\}$ of 5 values, count triples using all values in the window with both m and $m + 4$ present:

$$5^3 - 4^3 - 4^3 + 3^3 = 125 - 64 - 64 + 27 = 24$$

$$4 \times 24 = 96$$

ANSWER 96

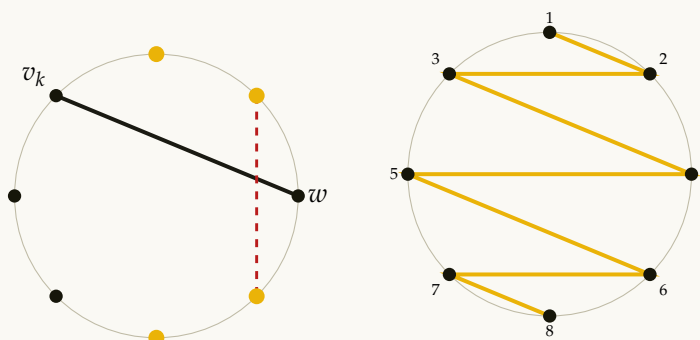
Q 18 Combinatorics

NONCROSSING PATHS

3 MARKS

Let P be a regular polygon with 8 vertices. By a labelling of P we mean an assignment of the integers $1, 2, \dots, 8$ to the vertices in some order. A labelling is considered *good* if the path consisting of line segments 1 to 2 , 2 to 3 , $\dots$, and 7 to 8 does not self-intersect. If N is the number of good labellings, what is the remainder when N is divided by 100?

SOLUTION



Left: taking w from the middle of the unused arc strands the gold vertices on both sides of the chord, and the red segment that must eventually join them crosses it. Right: a good labelling, where every new vertex is an end of the arc still unused.

Write the path as $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_8$, where v_k is the vertex labelled k , and let

$$U_k = \{v_{k+1}, v_{k+2}, \dots, v_8\}$$

be the set of vertices still unused after v_k .

The unused vertices always sit together on the circle. The eight vertices lie on a circle. A chord of a circle cuts the disc into two parts, and a segment joining a point of one part to a point of the other part must cross that chord.

Suppose U_k occupies one unbroken arc, and v_k is the vertex next to one of its two ends. If the next vertex v_{k+1} were taken from the middle of that arc, then the chord $v_k v_{k+1}$ would leave unused vertices on both of its sides. The path still has to visit all of them, so at some later stage it would join a vertex on one side to a vertex on the other side, and that segment would cross $v_k v_{k+1}$. The labelling would not be good.

So v_{k+1} must be one of the two end vertices of the arc U_k . Removing an end of an arc leaves an arc, and the new current vertex v_{k+1} is again next to an end of it, so the same statement holds at the next stage.

Starting the count. U_1 is the whole circle with v_1 removed, which is an arc, and v_1 is next to both of its ends. So the rule applies from the very first step.

Counting the choices.

v_1 : any of the 8 vertices

$v_2, v_3, \dots, v_7$: either end of the arc left over, 2 ways each

v_8 : one vertex remains, 1 way

Different choices give different labellings, and every good labelling arises exactly once this way.

$$N = 8 \times \underbrace{2 \times 2 \times 2 \times 2 \times 2 \times 2}_6 \times 1 = 8 \times 64 = 512$$

$$512 = 5 \times 100 + 12$$

The same count for a polygon with n vertices gives $n \times 2^{n-2}$, and the small cases can be checked by hand:

$$n = 3 : 3 \times 2 = 6 = 3! \quad (\text{every labelling of a triangle is good})$$

$$n = 4 : 4 \times 4 = 16 \quad \text{out of } 4! = 24$$

ANSWER 12

Q 19 Geometry · MIDPOINTS AND ANGLE GEOMETRY

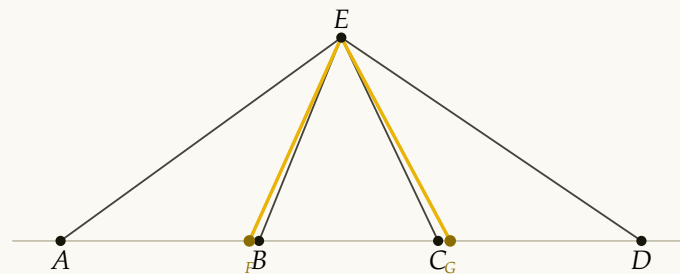
3 MARKS

Four points A, B, C , and D lie on a straight line, in this order. A point E , not on the line, satisfies

$$\angle AEB = \angle BEC = \angle CED = 45^\circ.$$

Let F and G be the midpoints of AC and BD , respectively. If $\angle FEG = x^\circ$, what is the value of x ?

SOLUTION



EF and EG are medians to the hypotenuses of AEC and BED .

$$\angle AEC = \angle AEB + \angle BEC = 90^\circ, \quad \angle BED = \angle BEC + \angle CED = 90^\circ$$

In right triangle AEC the median EF to the hypotenuse satisfies $EF = FA = FC$, so triangle FAE is isosceles:

$$\angle FEA = \angle FAE = \angle EAD = \alpha$$

In right triangle BED the median EG gives $EG = GB = GD$, so

$$\angle GED = \angle GDE = \angle EDA = \delta$$

In triangle AED ,

$$\angle AED = \angle AEB + \angle BEC + \angle CED = 135^\circ \implies \alpha + \delta = 45^\circ$$

$$\angle FEG = \angle AED - \angle AEF - \angle GED = 135^\circ - (\alpha + \delta) = 135^\circ - 45^\circ = 90^\circ$$

ANSWER 90

Q 20 Number Theory · CYCLIC DIGIT SHIFT

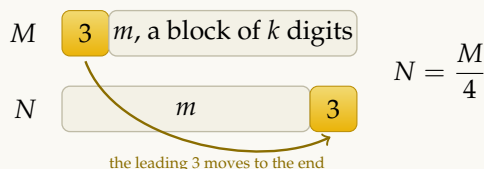
3 MARKS

Let M be the smallest positive integer with the following two properties:

- (a) The leading digit of M is equal to 3.
- (b) If N is the number obtained by moving this leading 3 to the units place and shifting all the other digits one place to the left, then $N = M/4$.

What is the sum of the digits of M ?

SOLUTION



M is the leading 3 followed by a block of k digits called m , and N is that same block followed by the 3. Written this way both numbers carry the same unknown.

Let M have $k + 1$ digits, so

$$M = 3 \times 10^k + m, \quad 0 \leq m < 10^k, \quad N = 10m + 3$$

$$4N = M \implies 40m + 12 = 3 \times 10^k + m$$

$$39m = 3 \times 10^k - 12 \implies 13m = 10^k - 4$$

So $10^k \equiv 4 \pmod{13}$:

k	1	2	3	4	5
$10^k \pmod{13}$	10	9	12	3	4

The smallest M comes from the smallest k , namely $k = 5$:

$$m = \frac{10^5 - 4}{13} = \frac{99996}{13} = 7692$$

$$M = 300000 + 07692 = 307692, \quad N = 076923 = 76923, \quad 4 \times 76923 = 307692$$

$$3 + 0 + 7 + 6 + 9 + 2 = 27$$

ANSWER 27

Section C

QUESTIONS 21 TO 30 · 5 MARKS EACH

Q 21 Algebra · COMPLEX-NUMBER SYSTEM

5 MARKS

Complex numbers x, y, z satisfy the following system of equations:

$$x^2 + y^2 + z = xy,$$

$$x + y^2 + z^2 = yz,$$

$$x^2 + y + z^2 = xz.$$

Determine the sum of all distinct possible values of

$$|(x^2 - y)(y^2 - z)(z^2 - x)|.$$

SOLUTION

Call the equations E_1, E_2, E_3 . Subtracting in pairs,

$$E_1 - E_2 : (x^2 - z^2) - (x - z) = y(x - z) \implies (x - z)(x + z - 1 - y) = 0$$

$$E_2 - E_3 : (y^2 - x^2) - (y - x) = z(y - x) \implies (y - x)(y + x - 1 - z) = 0$$

$$E_1 - E_3 : (y^2 - z^2) - (y - z) = x(y - z) \implies (y - z)(y + z - 1 - x) = 0$$

If x, y, z are all distinct, then

$$x + z = 1 + y, \quad x + y = 1 + z, \quad y + z = 1 + x$$

and adding the first two gives $2x = 2$, so $x = 1$; by symmetry $y = z = 1$, which contradicts distinctness.

If $x = y = z = t$, then E_1 gives

$$2t^2 + t = t^2 \implies t^2 + t = 0 \implies t = 0 \text{ or } t = -1$$

$$t = 0 : |(0)(0)(0)| = 0$$

$$t = -1 : |(-1)(-1)(-1)| = 1$$

If exactly two are equal, say $x = y \neq z$, then $(y - z)(y + z - 1 - x) = 0$ gives $y + z = 1 + x = 1 + y$, so $z = 1$, and E_1 becomes

$$x^2 + x^2 + 1 = x^2 \implies x^2 = -1 \implies x = \pm i$$

$$(x, y, z) = (i, i, 1) \text{ or } (-i, -i, 1)$$

$$(x^2 - y)(y^2 - z)(z^2 - x) = (-1 - i)(-1 - i)(1 - i) = (-1 - i)(1 - i)(-2) = (-2)(-2) = 4$$

The system is unchanged by the cycle $x \rightarrow y \rightarrow z \rightarrow x$, so the other four solutions of this type give the same value 4.

$$\{0, 4, 8\}, \quad 0 + 4 + 8 = 12$$

ANSWER 12

Q 22 Combinatorics · BLOCK PERMUTATIONS

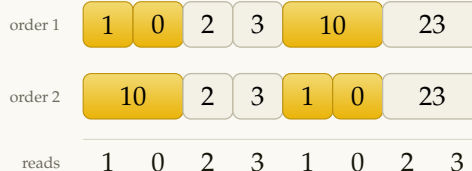
5 MARKS

Let N be the number of distinct 8-digit numbers obtained by arranging the six numbers

$$0, 1, 2, 3, 10, 23,$$

where the first digit of the 8-digit number is not zero. Find the sum of the digits of N .

SOLUTION



Two different orders of the six cards, and underneath, the one number they both read out. The gold cards are the ones that change places: the pair 1,0 and the single card 10 swap their jobs.

The six blocks carry $1 + 1 + 1 + 1 + 2 + 2 = 8$ digits. Arrangements of the blocks:

$$6! = 720$$

The first digit is 0 only when the block 0 comes first:

$$720 - 5! = 720 - 120 = 600$$

Two different arrangements can give the same string. The string contains the digits of the block 10 and also the separate blocks 1 and 0, so if the block 1 is immediately followed by the block 0, then exchanging that pair with the block 10 leaves the string unchanged. The same happens with 2, 3 and the block 23. These are the only repeats, since no other block pair spells another block.

$$\text{block 1 followed by block 0 : } 5! = 120 \quad (\text{none begins with 0})$$

$$\text{block 2 followed by block 3 : } 5! - 4! = 120 - 24 = 96$$

$$\text{both: } 4! = 24$$

Each single coincidence groups arrangements in twos and each double coincidence in fours:

$$\text{groups of 4 : } \frac{24}{4} = 6$$

$$\text{groups of 2 : } \frac{(120 - 24) + (96 - 24)}{2} = \frac{96 + 72}{2} = 84$$

$$\text{single arrangements : } 600 - 120 - 96 + 24 = 408$$

$$N = 408 + 84 + 6 = 498$$

$$4 + 9 + 8 = 21$$

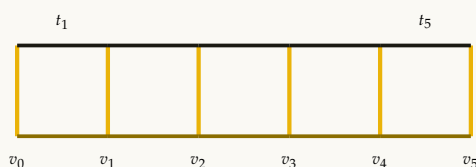
ANSWER 21

Q 23 Combinatorics · EDGE COLOURING

5 MARKS

A 1×5 rectangle is divided into five 1×1 squares by drawing four line segments parallel to the shorter side of the rectangle. Each of the resulting sixteen unit-length line segments is coloured red, blue, or green. A 1×1 square is called *colourful* if all the three colours are used in colouring its sides. If N is the number of ways of colouring such that all the five 1×1 squares are colourful, find the remainder when N is divided by 100.

SOLUTION



Sixteen segments: six vertical, five on top, five below.

Name the six vertical segments $v_0, v_1, \dots, v_5$ from left to right, and let t_i, b_i be the top and bottom segments of the i th square. The sides of the i th square are

$$v_{i-1}, v_i, t_i, b_i$$

Colour v_0 first: 3 ways. Now suppose v_{i-1} is already coloured, say with colour C . The square is colourful exactly when the triple (v_i, t_i, b_i) contains the other two colours A and B . Counting such triples,

$$3^3 - 2^3 - 2^3 + 1^3 = 27 - 8 - 8 + 1 = 12$$

This count does not depend on C , so each square contributes the same factor:

$$N = 3 \times 12^5 = 3 \times 248832 = 746496$$

$$746496 = 7464(100) + 96$$

ANSWER 96

Q 24 Geometry · RECTANGLE METRIC GEOMETRY

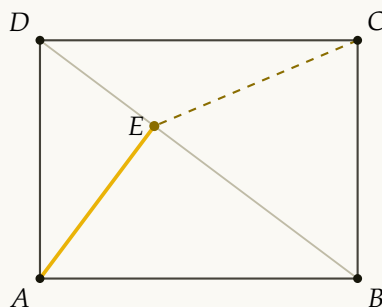
5 MARKS

Let $ABCD$ be a rectangle and let E be a point on BD such that $AE \perp BD$. If

$$AE = 12 \quad \text{and} \quad CE = \sqrt{193},$$

compute the area of the rectangle $ABCD$.

SOLUTION



AE is the altitude from A in triangle ABD ; CE is not perpendicular to BD .

Put $A = (0,0)$, $B = (p,0)$, $C = (p,q)$, $D = (0,q)$ and $d^2 = p^2 + q^2 = BD^2$. The foot of the perpendicular from A to BD is

$$E = \left(\frac{pq^2}{d^2}, \frac{p^2q}{d^2} \right), \quad AE = \frac{pq}{d} = 12$$

$$CE^2 = \left(p - \frac{pq^2}{d^2} \right)^2 + \left(q - \frac{p^2q}{d^2} \right)^2 = \left(\frac{p^3}{d^2} \right)^2 + \left(\frac{q^3}{d^2} \right)^2 = \frac{p^6 + q^6}{d^4}$$

$$p^6 + q^6 = (p^2 + q^2)^3 - 3p^2q^2(p^2 + q^2) = d^6 - 3p^2q^2d^2$$

$$CE^2 = d^2 - \frac{3p^2q^2}{d^2} = d^2 - 3 \left(\frac{pq}{d} \right)^2 = d^2 - 3(12)^2$$

$$193 = d^2 - 432 \implies d^2 = 625 \implies d = 25$$

$$pq = 12d = 12 \times 25 = 300$$

ANSWER 300

NOTE This paper reconstructs the question as printed; the area exceeds the two answer boxes, so record it as the number it is.

Q 25 Number Theory · UNIVERSAL DIVISIBILITY

5 MARKS

Let

$$E = \{ p^4 + p^2 - 2 \mid p \text{ is a prime, } p > 3 \}.$$

What is the largest positive integer that divides all the numbers in E ?

SOLUTION

$$p^4 + p^2 - 2 = (p^2 + 2)(p^2 - 1) = (p^2 + 2)(p - 1)(p + 1)$$

For a prime $p > 3$, p is odd and not a multiple of 3.

$$p - 1 \text{ and } p + 1 \text{ are consecutive even numbers} \implies 8 \mid (p - 1)(p + 1)$$

$$\begin{aligned}
 p &\not\equiv 0 \pmod{3} \implies 3 \mid (p-1)(p+1) \\
 p^2 &\equiv 1 \pmod{3} \implies p^2 + 2 \equiv 0 \pmod{3} \\
 \implies 8 \times 3 \times 3 &= 72 \text{ divides every member of } E
 \end{aligned}$$

Nothing larger works, because

$$\begin{aligned}
 p = 5 : 27 \times 4 \times 6 &= 648 = 2^3 \times 3^4 \\
 p = 7 : 51 \times 6 \times 8 &= 2448 = 2^4 \times 3^2 \times 17 \\
 \gcd(648, 2448) &= 2^3 \times 3^2 = 72
 \end{aligned}$$

ANSWER 72

Q 26 Combinatorics · GRAPH COLOURING

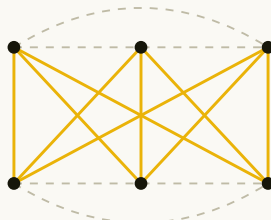
5 MARKS

There are n points in the plane, no three of which are collinear. Every pair of points is joined by a segment which is coloured red or blue such that the following conditions hold:

- (a) If A, B, C are three points such that AB is red and BC is blue, then AC is red.
- (b) For any point A , there are exactly three points B, C, D such that AB, AC, AD are red.

Find the sum of all possible values of n .

SOLUTION



$n = 6$: gold segments red, dashed segments blue.

Condition (b) says every point has exactly three red segments at it, so $n \geq 4$.

Fix a point A and let $R(A)$ be its three red neighbours. Take $B \in R(A)$ and any point C outside $\{A\} \cup R(A)$. Then AB is red and AC is blue, and applying (a) to the triple C, A, B gives CB red. So B is joined in red to all $n - 4$ such points C , and also to A :

$$3 \geq (n - 4) + 1 \implies n \leq 6$$

$n = 4$: all six segments red, every point has three red segments, and (a) is empty

$n = 5$: red degree 3 $\implies$ blue degree 1

so the blue segments would pair up all five points, which is impossible for an odd number of points.

$n = 6$: split the points as $\{P_1, P_2, P_3\}$ and $\{Q_1, Q_2, Q_3\}$

colour every $P_i Q_j$ red and every segment inside a group blue. Each point has exactly three red segments, and if AB is red then A, B are in different groups, so a blue BC puts C in B 's group, hence AC is red.

$$4 + 6 = 10$$

ANSWER 10

Q 27 Geometry · METRIC QUADRILATERAL

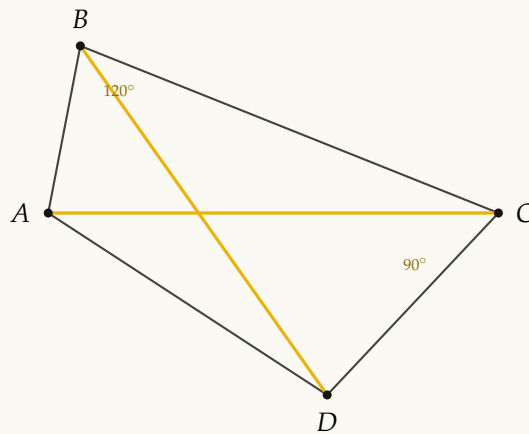
5 MARKS

The lengths of the sides of a convex quadrilateral are

$$\sqrt{a}, \quad \sqrt{a+3}, \quad \sqrt{a+2}, \quad \sqrt{2a+5},$$

in this order. The length of each diagonal is $\sqrt{2a+5}$. If θ° is the difference between the largest angle and the second largest angle of the quadrilateral, determine the value of θ .

SOLUTION



$a = 1$: both diagonals equal $\sqrt{7}$ and $\angle BCD = 90^\circ$.

Write $ABCD$ with

$$AB = \sqrt{a}, \quad BC = \sqrt{a+3}, \quad CD = \sqrt{a+2}, \quad DA = \sqrt{2a+5}, \quad AC = BD = \sqrt{2a+5}$$

In triangle BCD ,

$$BC^2 + CD^2 = (a+3) + (a+2) = 2a+5 = BD^2 \implies \angle BCD = 90^\circ$$

In triangle ACD , $AC = AD = \sqrt{2a+5}$, so dropping the perpendicular from A to CD bisects CD and

$$\cos \angle ACD = \frac{CD/2}{AC} = \frac{\sqrt{a+2}}{2\sqrt{2a+5}}, \quad \sin \angle ACD = \frac{\sqrt{7a+18}}{2\sqrt{2a+5}}$$

In triangle ABC , by the cosine rule,

$$\cos \angle ACB = \frac{AC^2 + BC^2 - AB^2}{2AC \cdot BC} = \frac{(2a+5) + (a+3) - a}{2\sqrt{(2a+5)(a+3)}} = \frac{a+4}{\sqrt{(2a+5)(a+3)}}$$

Convexity gives $\angle ACB + \angle ACD = \angle BCD = 90^\circ$, so $\cos \angle ACB = \sin \angle ACD$:

$$\begin{aligned}\frac{a+4}{\sqrt{(2a+5)(a+3)}} &= \frac{\sqrt{7a+18}}{2\sqrt{2a+5}} \implies \frac{2(a+4)}{\sqrt{a+3}} = \sqrt{7a+18} \\ 4(a+4)^2 &= (7a+18)(a+3) \implies 4a^2 + 32a + 64 = 7a^2 + 39a + 54 \\ 3a^2 + 7a - 10 &= 0 \implies (a-1)(3a+10) = 0 \implies a = 1 \\ AB = 1, \quad BC = 2, \quad CD = \sqrt{3}, \quad DA = \sqrt{7}, \quad AC = BD = \sqrt{7} \\ \cos \angle ABC &= \frac{1+4-7}{2(1)(2)} = -\frac{1}{2} \implies \angle ABC = 120^\circ \\ \cos \angle DAB &= \frac{1+7-7}{2(1)\sqrt{7}} = \frac{1}{2\sqrt{7}} \implies \angle DAB \approx 79.1^\circ \\ \angle BCD &= 90^\circ, \quad \angle CDA = 360^\circ - 120^\circ - 90^\circ - 79.1^\circ \approx 70.9^\circ \\ \theta &= 120 - 90 = 30\end{aligned}$$

ANSWER 30

Q 28 Number Theory · INTEGER RECURRENCES

5 MARKS

Let $a_1, a_2, \dots$ and $b_1, b_2, \dots$ be strictly increasing sequences of positive integers such that

- (a) $a_{n+1} = a_n + a_{n-1}$ for $n \geq 2$;
- (b) $b_n = 2b_{n-1}$ for all $n \geq 2$;
- (c) $a_{10} = b_{10} < 2026$.

Find the sum of all possible values of $a_1 + b_1$.

SOLUTION

Iterating the first rule,

$$a_3 = a_1 + a_2, \quad a_4 = a_1 + 2a_2, \quad a_5 = 2a_1 + 3a_2, \quad \dots, \quad a_{10} = 21a_1 + 34a_2$$

$$b_{10} = 2^9 b_1 = 512 b_1$$

$$512 b_1 < 2026 \implies b_1 \leq 3$$

The sequence a is strictly increasing, so $a_2 > a_1 \geq 1$. Solve $21a_1 + 34a_2 = 512b_1$ modulo 21, using $34 \equiv 13$ and $13 \times 13 = 169 \equiv 1$:

$$13a_2 \equiv 512b_1 \pmod{21} \implies a_2 \equiv 13 \times 512b_1 \pmod{21}$$

$$b_1 = 1: 512 \equiv 8, \quad a_2 \equiv 13(8) = 104 \equiv 20, \quad a_2 = 20 \implies 34(20) = 680 > 512 \quad \text{no solution}$$

$$b_1 = 2: 1024 \equiv 16, \quad a_2 \equiv 13(16) = 208 \equiv 19, \quad a_2 = 19 \implies 21a_1 = 1024 - 646 = 378, \quad a_1 = 18$$

$$18 < 19 \implies a_1 + b_1 = 18 + 2 = 20$$

$$b_1 = 3: 1536 \equiv 3, \quad a_2 \equiv 13(3) = 39 \equiv 18$$

$$a_2 = 18 \implies 21a_1 = 1536 - 612 = 924, \quad a_1 = 44 > 18 \quad \text{rejected}$$

$$a_2 = 39 \implies 21a_1 = 1536 - 1326 = 210, \quad a_1 = 10 < 39 \implies a_1 + b_1 = 10 + 3 = 13$$

$$20 + 13 = 33$$

ANSWER 33

Q 29 Number Theory · MODULAR EXPONENTIATION

5 MARKS

Let

$$n = \frac{4^{31} - 1}{3}.$$

Find the remainder when 2^{n-1} is divided by n .

SOLUTION

$$n = \frac{2^{62} - 1}{3} \implies 3n = 2^{62} - 1 \implies n \mid 2^{62} - 1 \implies 2^{62} \equiv 1 \pmod{n}$$

So it is enough to show that $62 \mid n - 1$.

$$n - 1 = \frac{2^{62} - 1 - 3}{3} = \frac{2^{62} - 4}{3} = \frac{4(2^{60} - 1)}{3}$$

The factor 4 supplies the 2 in $62 = 2 \times 31$. For the 31:

$$2^5 - 1 = 31 \quad \text{and} \quad 5 \mid 60 \implies 2^5 - 1 \mid 2^{60} - 1 \implies 31 \mid 2^{60} - 1$$

$$\gcd(31, 3) = 1 \implies 31 \mid \frac{2^{60} - 1}{3}$$

$$\implies 62 \mid n - 1, \quad n - 1 = 62k$$

$$2^{n-1} = (2^{62})^k \equiv 1^k = 1 \pmod{n}$$

ANSWER 1

Q 30 Geometry · ANGLE CHASING

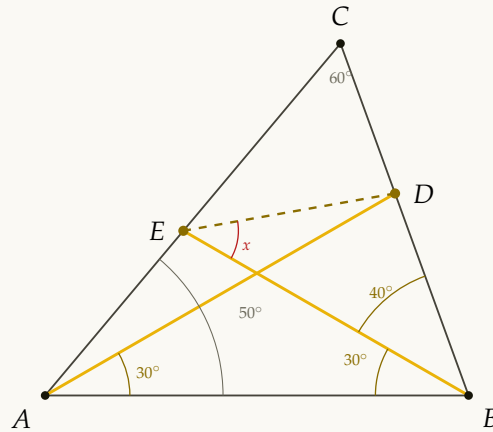
5 MARKS

In triangle ABC , it is given that $\angle CAB = 50^\circ$ and $\angle ABC = 70^\circ$. Points D and E are chosen on sides BC and AC , respectively, such that

$$\angle ABE = \angle DAB = 30^\circ.$$

If $\angle DEB = x^\circ$, what is the value of x ?

SOLUTION



D lies on BC and E on AC . The two gold cevians make $\angle DAB = \angle ABE = 30^\circ$, which leaves $\angle DBE = 40^\circ$, and x is the angle the dashed segment DE makes with EB .

$$\angle BCA = 180^\circ - 50^\circ - 70^\circ = 60^\circ$$

Take $AB = 1$. In triangle ABD , with D on BC ,

$$\angle DAB = 30^\circ, \quad \angle ABD = 70^\circ, \quad \angle ADB = 80^\circ$$

$$\frac{BD}{\sin 30^\circ} = \frac{AB}{\sin 80^\circ} \implies BD = \frac{\sin 30^\circ}{\sin 80^\circ}$$

In triangle ABE , with E on AC ,

$$\angle ABE = 30^\circ, \quad \angle BAE = 50^\circ, \quad \angle AEB = 100^\circ$$

$$BE = \frac{\sin 50^\circ}{\sin 100^\circ}$$

$$\frac{BE}{BD} = \frac{\sin 50^\circ}{\sin 100^\circ} \cdot \frac{\sin 80^\circ}{\sin 30^\circ} = \frac{\sin 50^\circ}{\sin 30^\circ} = 2 \sin 50^\circ$$

In triangle BDE ,

$$\angle DBE = \angle ABC - \angle ABE = 70^\circ - 30^\circ = 40^\circ$$

$$\angle BDE + \angle BED = 140^\circ$$

$$\frac{BD}{\sin \angle BED} = \frac{BE}{\sin \angle BDE} \implies \frac{\sin \angle BDE}{\sin \angle BED} = \frac{BE}{BD} = 2 \sin 50^\circ$$

Putting $\angle BED = x^\circ$ and $\angle BDE = (140 - x)^\circ$,

$$\frac{\sin(140^\circ - x^\circ)}{\sin x^\circ} = 2 \sin 50^\circ$$

$$x = 40 : \quad \frac{\sin 100^\circ}{\sin 40^\circ} = \frac{\sin 80^\circ}{\sin 40^\circ} = \frac{2 \sin 40^\circ \cos 40^\circ}{\sin 40^\circ} = 2 \cos 40^\circ = 2 \sin 50^\circ$$

This is the only solution, because

$$\frac{\sin(140^\circ - x^\circ)}{\sin x^\circ} = \frac{\sin 140^\circ \cos x^\circ - \cos 140^\circ \sin x^\circ}{\sin x^\circ} = \sin 140^\circ \cot x^\circ - \cos 140^\circ$$

and $\cot x^\circ$ strictly decreases as x runs from 0 to 140, while $\sin 140^\circ > 0$.

ANSWER 40

[ANSWER KEY]

All thirty answers on *one page*

Mark your attempt against this before you read the working. A wrong answer is worth more than a right one here, because it tells you which question to read first.

Q1 41	Q2 63	Q3 52	Q4 12	Q5 50
Q6 3	Q7 31	Q8 5	Q9 2	Q10 13
Q11 2	Q12 42	Q13 18	Q14 11	Q15 51
Q16 15	Q17 96	Q18 12	Q19 90	Q20 27
Q21 12	Q22 21	Q23 96	Q24 300	Q25 72
Q26 10	Q27 30	Q28 33	Q29 1	Q30 40

Question 24 asks for an area, and that area is 300, which is larger than the two answer boxes allow. Every other answer is an integer from 00 to 99.

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