

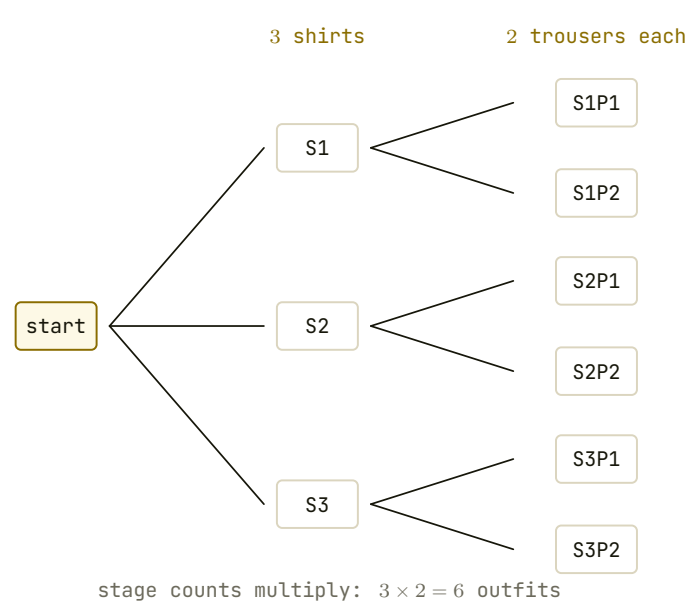
CONCEPT MAP

IOQM · COMBINATORICS

5 TOPICS · 38 CORE IDEAS · ONE SHEET

A COUNTING PRINCIPLES 01-08

81 ADDITION AND MULTIPLICATION PRINCIPLES A-01



- Addition:** If a task is done in one of several distinct ways, with m ways of one kind and n of the other, there are $m + n$ ways in all.
- Multiplication:** If a task is done in stages, and stage i always offers n_i choices whatever happened before, the stages together give

KEY

$$N = n_1 \cdot n_2 \cdots n_k$$

- What must not depend on earlier choices is the **number** of options at each stage, not the options themselves.
- "Or" with disjoint cases adds; "and then" multiplies.

Divisors of 72 = $2^3 \cdot 3^2$: choose the power of 2 in 4 ways, then the power of 3 in 3 ways, giving 12.

Overlapping cases do not simply add. When the cases share outcomes, adding counts the shared ones twice; inclusion and exclusion (section 17) is the correction.

82 PERMUTATIONS A-02

KEY

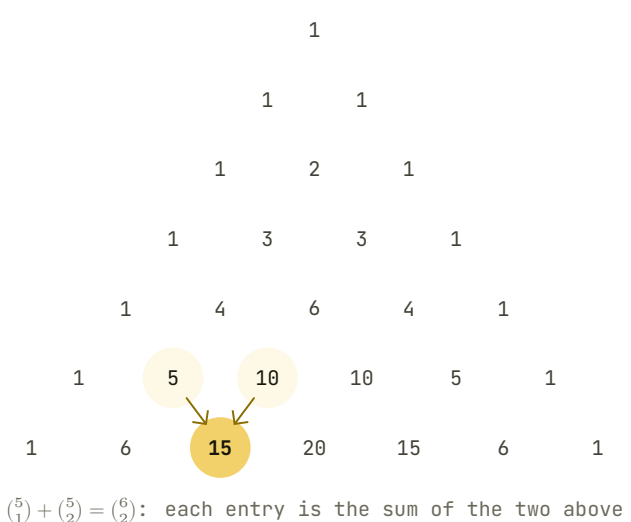
$$P(n, r) = \frac{n!}{(n-r)!}$$

- n distinct objects in a row: $n!$ ways. An ordered choice of r of them: $P(n, r)$.
- With repetition allowed, r places each filled from n options: n^r ways.
- Objects that are alike:** n objects with n of one kind, k of another, and so on, can be arranged in $\frac{n!}{a!b!\cdots}$ ways.

BANANA has 3 A, 2 N and 1 B: $\frac{6!}{3!2!1!} = 60$ arrangements.

0! = 1 is a convention, and the right one. It makes $P(n, n) = n!/0!$ and $\binom{n}{0} = 1$ come out correctly without special cases.

83 COMBINATIONS AND PASCAL'S TRIANGLE A-03



$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$: each entry is the sum of the two above

KEY

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

- Symmetry:** $\binom{n}{r} = \binom{n}{n-r}$. **Pascal:** $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$.
- $\sum_{k=0}^n \binom{n}{k} = 2^n$, and for $n \geq 1$ the even and odd terms each sum to 2^{n-1} .
- Hockey stick:** $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$.
- Vandermonde:** $\sum_{k=0}^r \binom{n}{k} \binom{m}{r-k} = \binom{n+m}{r}$.

Prove identities with a story, not algebra. Pascal's rule says a chosen group either contains a particular person or it does not. Most identities on this sheet fall out of one such sentence.

84 CIRCULAR ARRANGEMENTS A-04

KEY

$$n \text{ distinct objects round a circle: } (n-1)!$$

- Rotations count as the same, so fix one object and arrange the rest in a line.
- If turning over is also allowed, as for a necklace or garland with $n \geq 3$, there are $\frac{(n-1)!}{2}$.
- If the seats are labelled, it is a line again: $n!$.

5 boys and 5 girls alternating round a table: seat the boys in 4! ways, then the girls in the 5 gaps in 5! ways, 2880 in all.

Dividing by n needs every object distinct. With repeated objects, a symmetric pattern has fewer rotations. Two red and two blue beads give $\binom{6}{2} = 6$ strings but only 2 necklaces up to rotation, not 6/4.

B COMPLEMENTARY COUNTING A-05

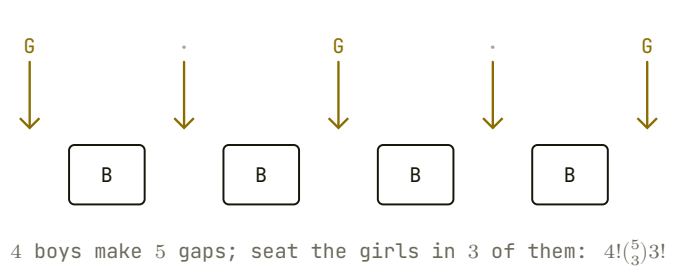
KEY

$$\#(\text{at least one}) = \#(\text{all}) - \#(\text{none})$$

- Count what you **do not** want, and subtract it from the total.
- Use it whenever the complement is one simple case while the original splits into many.

Four digit numbers with at least one repeated digit: $9000 - 9 \cdot 9 \cdot 8 \cdot 7 = 9000 - 4536 = 4464$.

86 TOGETHER, AND NEVER TOGETHER A-06



- Together:** glue the group of k into one block, arrange the $n - k + 1$ units, then arrange inside the block: $(n - k + 1)k!$.
- No two together:** arrange the other p objects first; they leave $p + 1$ gaps, and each of the m separated objects takes its own gap:

KEY

$$p! \binom{p+1}{m} m!$$

4 boys and 3 girls in a row with no two girls adjacent: $4! \cdot \binom{5}{3} \cdot 3! = 24 \cdot 10 \cdot 6 = 1440$.

"Not all together" is not "no two together". The first is the total minus the all-together count; the second is the gap method. They are different numbers.

87 NUMBERS WITH DIGIT CONDITIONS A-07

- The leading digit cannot be 0: fill the first place first, or count all strings and subtract those that start with 0.
- For a divisibility condition, fix the deciding digits first: the last digit for 2 or 5, the last two for 4.
- Sum of all numbers** formed by arranging n distinct non zero digits: each digit sits in each place $(n - 1)!$ times, so

KEY

$$(n-1)! \left(\sum d \right) \frac{11 \cdots 1}{n}$$

With 1, 2, 3: $2! \cdot 6 \cdot 111 = 1332$, and indeed $123 + 132 + 213 + 321 + 312 + 231 = 1332$.

88 OVERCOUNTING AND DIVIDING A-08

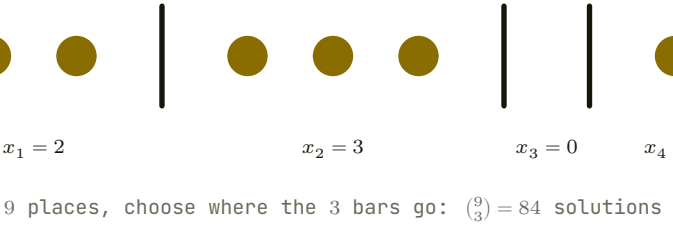
- Count in a convenient ordered way, then divide by k if **every** object has been counted exactly k times.
- Handshakes among n people: $n(n-1)$ ordered pairs, each handshake counted twice, so $\binom{n}{2}$.
- When objects are counted different numbers of times, sort them by multiplicity and divide each group by its own factor.

IOQM 2026, Q22: the cards 0, 1, 2, 3, 10, 23 give 600 orders without a leading 0, but 1 then 0 reads like the card 10, and 2 then 3 like 23. Strings come from 1, 2 or 4 orders, giving $408 + 48 + 6 = 468$.

Check the multiplicity before dividing. Dividing a count by a single factor is only valid when every object is overcounted equally; Q22 is built to punish that step.

B DISTRIBUTIONS & BIJECTIONS 09-16

89 STARS AND BARS B-09



KEY

$$\# \{x_i \geq 0\} = \binom{n+k-1}{k-1}$$
$$\# \{x_i \geq 1\} = \binom{n-1}{k-1}$$

- These count integer solutions of $x_1 + \cdots + x_k = n$; equally, ways to put n identical objects into k distinct boxes.
- A lower bound $x_i \geq a_i$ is removed by substituting $y_i = x_i - a_i$.
- An inequality $x_1 + \cdots + x_k \leq n$ becomes an equation with one extra slack variable: $\binom{n+1}{k}$ non negative solutions.

$x + y + z = 10$ in non negative integers: $\binom{12}{2} = 66$.

90 UPPER BOUNDS BY INCLUSION AND EXCLUSION B-10

For $x_1 + \cdots + x_k = n$ with $0 \leq x_i \leq a_i$, subtract the solutions in which some variable is at least $c + 1$:

KEY

$$\sum_{j=0}^k (-1)^j \binom{k}{j} \binom{n - j(c+1) + k - 1}{k-1}$$

- Stop when $n - j(c+1)$ becomes negative.
- The same number is the coefficient of x^n in $(1 + x + \cdots + x^a)^k$.

Three dice showing a total of 10: put $y_i = x_i - 1$, so $y_1 + y_2 + y_3 = 7$ with $y_i \leq 5$: $\binom{9}{2} - 3\binom{7}{2} = 36 - 9 = 27$.

C DISTINCT OBJECTS INTO BOXES B-11

- n distinct objects into k distinct boxes: k^n ways, the number of functions from an n set to a k set.
- At most one per box needs $n \leq k$ and gives $P(k, n)$ ways.
- Every box non empty** (onto functions):

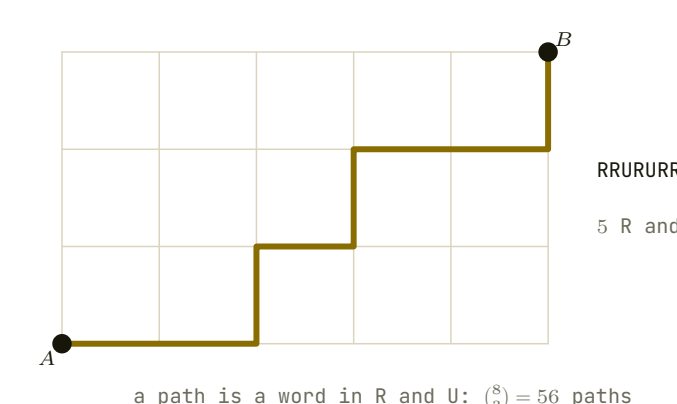
KEY

$$\sum_{j=0}^k (-1)^j \binom{k}{j} (k-j)^n$$

- Into k **identical** non empty boxes: divide the onto count by $k!$. This is the Stirling number $S(n, k)$.
- $S(n, 2) = 2^{n-1} - 1$ and $S(n, n-1) = \binom{n}{2}$.

Onto maps from 5 objects to 3 boxes: $3^5 - 3 \cdot 2^5 + 3 \cdot 1^5 = 243 - 96 + 3 = 150$, so $S(5, 3) = 25$.

92 LATTICE PATHS B-12



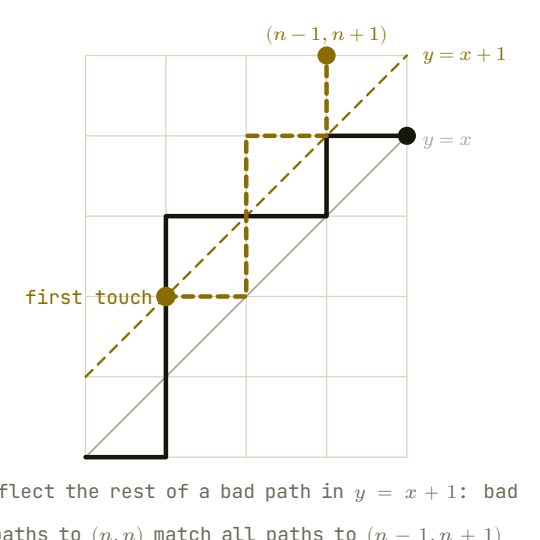
Paths from $(0, 0)$ to (m, n) with unit steps right and up are words with m R and n U:

KEY

$$\binom{m+n}{m}$$

- Through a point P :** multiply the counts to P and from P . **Avoiding P :** subtract that from the total.
- With obstacles or extra steps, fill in the grid: each point gets the sum of the counts at the points that lead into it.

93 CATALAN NUMBERS AND REFLECTION B-13



Paths from $(0, 0)$ to (n, n) that never go above $y = x$ number

KEY

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

- Reflection:** a bad path first touches $y = x + 1$; reflecting the rest of it in that line gives, one to one, every path to $(n-1, n+1)$.
- $C_0, C_1, \dots = 1, 1, 2, 5, 14, 42, 132, 429$, and $C_{n+1} = \sum_{k=0}^n C_k C_{n-k}$.
- C_k also counts balanced strings of n pairs of brackets, triangulations of a convex $(n+2)$ gon, and ways to bracket a product of $n+1$ factors.

94 BIJECTIONS B-14

- To count a set, match it one to one with a set that is easier to count.
- Subsets of an n set match binary strings of length n : 2^n .
- Choices of r objects from n types with repetition match stars and bars: $\binom{n+r-1}{r-1}$; so do non decreasing maps $\{1, \dots, r\} \rightarrow \{1, \dots, n\}$.
- No two consecutive:** r numbers from $1, \dots, n$ with no two consecutive match, by subtracting $i-1$ from the i th smallest, arbitrary r subsets of $1, \dots, n-r+1$:

KEY

$$\binom{n-r+1}{r}$$

A bijection needs both directions. Describe the map, describe its inverse, and check that each undoes the other. A map that is only onto, or only one to one, proves an inequality, not an equality.

95 COMPOSITIONS B-15

A **composition** of n is an ordered sum of positive parts. There are $\binom{n-1}{k-1}$ with k parts, so in total

KEY

$$\sum_k \binom{n-1}{k-1} = 2^{n-1}$$

- Restricted parts:** if the parts must come from a set P , split on the first part: $g(n) = \sum_{p \in P} g(n-p)$, with $g(0) = 1$ and $g(n) = 0$ for $n < 0$.
- Compositions into 1s and 2s are counted by the Fibonacci numbers.

IOQM 2026, Q7: the digits form blocks 2, 4, 8, 8; a parity argument rules out 8!, and halving leaves compositions of 7 into parts 1, 2, 4, giving $g(7) = 31$.

D PARTITIONS B-16

- A **partition** of n is an unordered sum of positive parts. List them with parts in decreasing order, so that each is written once.
- $p(1), \dots, p(10) = 1, 2, 3, 5, 7, 11, 15, 22, 30, 42$.
- Partitions into at most k parts are as many as partitions into parts of size at most k ; flip the dot diagram across its diagonal.
- Euler:** partitions into distinct parts are as many as partitions into odd parts.

$n = 6$: distinct parts 6, 5+1, 4+2, 3+2+1; odd parts 5+1, 3