

CONCEPT MAP

IOQM · GEOMETRY

5 TOPICS · 38 CORE IDEAS · ONE SHEET



A ANGLES & TRIANGLES 01-05

01 ANGLES AND PARALLEL LINES A-01

with $l_1 \parallel l_2$, corresponding and alternate angles are equal, co interior angles add to 180°

- Vertically opposite angles are equal; angles on a straight line add to 180° , and angles round a point to 360° .
- A transversal of two parallel lines makes corresponding angles equal, alternate angles equal, and co interior angles supplementary.
- Each of these is also a **test**: if any one holds, the two lines are parallel.

Parallels are free equal angles. In an angle chase, mark every angle that a pair of parallel lines hands you before doing anything else; the rest of the figure usually follows.

02 ANGLE SUM AND EXTERIOR ANGLE A-02

the exterior angle at C equals the sum of the two interior angles opposite it

KEY

$\angle A + \angle B + \angle C = 180^\circ$

- An exterior angle equals the sum of the two interior angles opposite it, so it is larger than each of them.
- A convex n -gon has interior angles adding to $(n-2) \cdot 180^\circ$ and exterior angles adding to 360° .

Angles in the ratio 2 : 3 : 4 are $40^\circ, 60^\circ, 80^\circ$.

03 ISOSCELES AND SPECIAL TRIANGLES A-03

side ratios 1 : 1 : $\sqrt{2}$ and 1 : $\sqrt{3}$: 2

- Two sides are equal **if and only if** the angles opposite them are equal.
- In an isosceles triangle, the median, altitude and angle bisector from the apex all lie on the perpendicular bisector of the base.
- An isosceles triangle with one angle of 60° is equilateral.

KEY

equilateral, side a : $h = \frac{\sqrt{3}}{2}a$, $\Delta = \frac{\sqrt{3}}{4}a^2$
 $R = \frac{a}{\sqrt{3}}$, $r = \frac{a}{2\sqrt{3}}$

Look for the 30, 60, 90 triangle. Any 30° or 60° angle next to a right angle gives the ratio 1 : $\sqrt{3}$: 2 at once, and a 45° angle gives 1 : 1 : $\sqrt{2}$.

04 CONGRUENCE A-04

TEST	MUST MUST MATCH
SSS	all three sides
SAS	two sides and the angle between them
ASA, AAS	two angles and any corresponding side
RHS	right angle, hypotenuse and one other side

- SSA is **not** a test in general: two sides and a non included angle can fit two different triangles. It is safe only when the angle is opposite the longer given side; RHS is the commonest such case.
- Congruent triangles give equal lengths and equal angles everywhere; that is what they are used for.

The congruent pair is usually hidden by a symmetry. Look for a reflection in an axis or a rotation about a point that carries the figure to itself; the triangles it swaps are the congruent ones.

05 TRIANGLE INEQUALITY AND SIDE ORDER A-05

KEY

$|b - c| < a < b + c$

- Positive lengths a, b, c form a triangle exactly when each is less than the sum of the other two. It is enough to check the largest.
- The larger side is opposite the larger angle, and conversely.
- The triangle inequality is also an extremal tool: a broken path between two points is never shorter than the straight one.

Triangles with integer sides and perimeter 12: (2, 5, 5), (3, 4, 5), (4, 4, 4), all three.

06 PYTHAGORAS AND THE ANGLE TEST A-06

KEY

$\angle C$ is acute, right, obtuse $\Leftrightarrow c^2$ is $<$, $=$, $>$ $a^2 + b^2$

- The **converse** of Pythagoras holds: if $a^2 + b^2 = c^2$ then the angle opposite c is a right angle.
- IOQM 2026, Q27: $BC^2 + CD^2 = (a+3)^2 + (a+2)^2 = 2a+5 = BD^2$ gives $\angle BCD = 90^\circ$ before any other work.
- A triangle is acute exactly when the square of its longest side is less than the sum of the other two squares.

Sides 7, 8, 9: $8^2 + 9^2 < 7^2 + 8^2 + 9^2$, so the triangle is acute.

07 THE MEDIAN TO THE HYPOTENUSE A-07

$MA = MB = MC$; the midpoint of the hypotenuse is the circumcentre

- In a right triangle, the midpoint of the hypotenuse is the same distance from all three vertices: the median to the hypotenuse is half of it.
- So the hypotenuse is a diameter of the circumcircle; this is Thales' theorem read backwards.
- Converse**: if the median to a side equals half that side, the angle opposite it is right.

IOQM 2026, Q19: EF and EG are medians to the hypotenuses of the right triangles AEC and BED , so FAE and GDE are isosceles, the base angles add to 45° , and $\angle FEG = 135^\circ - 45^\circ = 90^\circ$.

08 THE MIDPOINT THEOREM A-08

$MN \parallel BC$ and $MN = \frac{1}{2}BC$

KEY

- Converse**: the line through the midpoint of one side, parallel to a second side, bisects the third.
- The three midpoints form the medial triangle, similar to the original with ratio $\frac{1}{2}$ and a quarter of its area.
- Varignon**: the midpoints of the sides of any quadrilateral form a parallelogram, possibly flat, with half the area whenever the quadrilateral is not self crossing.

09 SIMILARITY & LENGTHS 09-16

KEY

$\frac{[ABC]}{[A'B'C']} = k^2$ when the sides are in ratio k

A line parallel to the base, a third of the way down from the apex, cuts off a triangle with $\frac{1}{9}$ of the area.

Write similar triangles with matched vertex order. $\triangle ABC \sim \triangle DEF$ means $A \leftrightarrow D, B \leftrightarrow E, C \leftrightarrow F$, and the ratios $AB/DE = BC/EF = CA/FD$ can then be read off without looking at the figure.

10 BASIC PROPORTIONALITY 0-10

$DE \parallel BC \Leftrightarrow \frac{AD}{DB} = \frac{AE}{EC}$, and then $\triangle ADE \sim \triangle ABC$

KEY

$DE \parallel BC \Leftrightarrow \frac{AD}{DB} = \frac{AE}{EC}$

- Intercept theorem**: parallel lines cut any two transversals in the same ratio.
- In a trapezium with parallel sides a and b , the segment through the meeting point of the diagonals, parallel to them, has length $\frac{2ab}{a+b}$.

11 ANGLE BISECTOR THEOREM 0-11

$BD : DC = AB : AC = c : b$

KEY

$\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$

- The external bisector at A divides BC externally in the same ratio $c : b$, when $b \neq c$.
- Bisector length**: $AD^2 = bc(1 - (\frac{a^2}{b^2 + c^2}))$.
- The incentre divides the bisector AD in the ratio $AI : ID = (b+c) : a$.

12 THE ALTITUDE TO THE HYPOTENUSE 17-26

$h^2 = p \cdot q$, $b^2 = p \cdot c$, $a^2 = q \cdot c$, $h = \frac{ab}{c}$

KEY

$h^2 = p \cdot q$, $b^2 = p \cdot c$, $a^2 = q \cdot c$, $h = \frac{ab}{c}$

- The altitude splits the right triangle into two triangles similar to it and to each other, which gives all four relations.
- $h = ab/c$ is the area computed two ways.

IOQM 2026, Q24: In a $p \times q$ rectangle with diagonal d , the altitude from A to BD is $AE = pq/d$, and coordinates give $CE^2 = d^2 - 3AE^2$. With $AE = 12$, $CE^2 = 193$: $d = 25$, area $pq = 300$.

13 MEDIANS: APOLLONIUS AND STEWART 0-13

$b^2m + c^2n = a(d^2 + mn)$, with $a = m + n$

KEY

$m^2 = \frac{2b^2 + 2c^2 - a^2}{4}$

- Stewart**: a cevian $AD = d$ with $BD = m$, $DC = n$ satisfies $b^2m + c^2n = a(d^2 + mn)$; Apollonius is the case $m = n$.
- The centroid divides each median 2 : 1 from the vertex, and the three medians cut the triangle into six pieces of equal area.
- Parallelogram law**: the sum of the squares of the four sides equals the sum of the squares of the diagonals.

14 AREA FORMULAS 0-14

KEY

$\Delta = \frac{1}{2}ab \sin C = \sqrt{s(s-a)(s-b)(s-c)}$
 $\Delta = \frac{abc}{4R}$

- s is the semiperimeter; $\sqrt{s(s-a)(s-b)(s-c)}$ is Heron's formula.
- Triangles with the same height have areas in the ratio of their bases. Triangles on the same base between the same parallels have equal areas.
- A cevian from A to BC splits the area of the triangle $DD : DC$.

Sides 13, 14, 15: $s = 21$, $\Delta = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$, so $r = 84/21 = 4$ and $R = \frac{13 \cdot 14 \cdot 15}{4 \cdot 84} = \frac{65}{8}$.

15 CEVA AND MENELAUS 0-15

concurrent exactly when $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$

KEY

$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$

- Ceva**: for D, E, F on the sides BC, CA, AB , the cevians AD, BE, CF are concurrent exactly when the product is 1.
- Menelaus**: for D, E, F on the lines BC, CA, AB , they are collinear exactly when the same product of signed ratios is -1; with plain lengths it is 1, and the line meets two sides internally and the third externally, or none internally.
- Trigonometric Ceva**: $\frac{\sin \angle BAD}{\sin \angle DAC} \cdot \frac{\sin \angle CBE}{\sin \angle EBA} \cdot \frac{\sin \angle ACF}{\sin \angle FCB} = 1$, the form to use when the data are angles.

16 TRIGONOMETRY IN THE TRIANGLE 0-16

KEY

$\frac{a}{\sin A} = 2R$, $a^2 = b^2 + c^2 - 2bc \cos A$

ANGLE	SIN	COS	TAN
15°	$\frac{\sqrt{6} - \sqrt{2}}{4}$	$\frac{\sqrt{6} + \sqrt{2}}{4}$	$2 - \sqrt{3}$
18°	$\frac{\sqrt{5} - 1}{4}$	$\frac{\sqrt{10 + 2\sqrt{5}}}{4}$	
36°	$\frac{\sqrt{10 - 2\sqrt{5}}}{4}$	$\frac{\sqrt{5} + 1}{4}$	
75°	$\frac{\sqrt{6} + \sqrt{2}}{4}$	$\frac{\sqrt{6} - \sqrt{2}}{4}$	$2 + \sqrt{3}$

- IOQM 2026, Q13: with $CA = CB = a$, $CD = a/\tan 75^\circ = a(2 - \sqrt{3})$; the area $[ACD] = \frac{1}{2}a \cdot a(2 - \sqrt{3}) = 81$ fixes a^2 , and $BD = a(\sqrt{3} - 1) = 18$.
- IOQM 2026, Q8: the cosine rule in ACD and then in ABC gives the angles $60, 30, 150, 120$ degrees, so $x/y = 5$.

17 CIRCLES 17-26

17 THE INSCRIBED ANGLE THEOREM C-17

angle at the centre is twice the angle at the circumference

KEY

$\angle BOC = 2\angle BAC$ (A on the major arc)

- Angles in the same segment are equal, and the angle in a semicircle is a right angle.
- An inscribed angle is half the arc it cuts off; in one circle, or equal circles, equal chords cut off equal arcs and subtend equal angles from corresponding arcs.
- If A lies on the minor arc BC , the inscribed angle is 180° minus the angle from the major arc.

18 ARC MIDPOINTS C-18

N : midpoint of arc BAC
 $NN' \perp BC$
 N' : midpoint of arc BC

KEY

$AC \cdot BD = AB \cdot CD + AD \cdot BC$

- The perpendicular bisector of a chord passes through the centre and meets the circle at the midpoints of both arcs.
- The bisector of $\angle A$ meets the circumcircle again at the midpoint of the arc BC not containing A ; the external bisector meets it at the other arc midpoint.
- Incentre lemma**: if I is the midpoint of the arc BC not containing A of the circumcircle of an equilateral triangle ABC , then $PA = PB + PC$.

IOQM 2026, Q5: arc BC away from A is $2 \cdot 80^\circ = 160^\circ$, so arc BAC is 200° ; N halves it, and $\angle NBC = \frac{1}{2} \cdot 100^\circ = 50^\circ$.

A regular pentagon with side 1 and diagonal d : Ptolemy on four vertices gives $d^2 = d + 1$, so $d = \frac{1+\sqrt{5}}{2}$.

19 CYCLIC QUADRILATERALS C-19

opposite angles add to 180°

KEY

$\angle A + \angle C = \angle B + \angle D = 180^\circ$

- An exterior angle of a cyclic quadrilateral equals the interior angle opposite it.
- Tests for four points to be concyclic**: a pair of opposite angles is supplementary; or a segment subtends equal angles at two points on the same side of it; or $PA \cdot PB = PC \cdot PD$ where lines AD and CB meet at P , with P inside both segments or outside both.
- A parallelogram is cyclic only when it is a rectangle; an isosceles trapezium is always cyclic.

20 TANGENTS C-20

tangent $\perp$ radius; the tangent and chord PA make the angle in the alternate segment

- A tangent is perpendicular to the radius at the point of contact.
- The two tangents from an outside point are equal, and the centre lies on the bisector of the angle between them.
- Alternate segment**: the angle between a tangent and a chord equals the inscribed angle on the far side of the chord.

KEY

tangent length from $P = \sqrt{OP^2 - r^2}$

Pitot: a convex quadrilateral has an incircle exactly when $AB + CD = BC + DA$; one direction is equal tangents added round the figure.

21 POWER OF A POINT C-21

$PA \cdot PB = PC \cdot PD$, $QE^2 = QF \cdot QP$

KEY

$PA \cdot PB = PC \cdot PD = |OP^2 - r^2|$

- It holds for two chords through an inside point, and for two secants from an outside point.
- For a tangent QT and a secant QXY from the same outside point, $QT^2 = QX \cdot QY$.
- The **radical axis** of two non concentric circles, the points with equal power to both, is a straight line perpendicular to the line of centres.

Chords meet with $PA = 4$, $PB = 6$, $PC = 3$: then $PD = \frac{9}{2} = 4.5$.

22 TWO CIRCLES C-22

direct: $\sqrt{d^2 - (r_1 - r_2)^2}$
transverse: $\sqrt{d^2 - (r_1 + r_2)^2}$, where $d = O_1O_2$

DISTANCE OF CENTRES	POSITION	COMMON TANGENTS
$d > r_1 + r_2$	apart	4
$d = r_1 + r_2$	touching outside	3
$ r_1 - r_2 < d < r_1 + r_2$	cutting	2
$d = r_1 - r_2 > 0$	touching inside	1
$d < r_1 - r_2 $	one inside the other	0

Touching circles: the two centres and the point of contact lie on one line. That single fact turns most problems about chains of tangent circles into right triangles with sides $r_1 + r_2$ or $r_1 - r_2$.

23 PTOLEMY'S THEOREM C-23

$AC \cdot BD = AB \cdot CD + AD \cdot BC$

KEY

$AC \cdot BD = AB \cdot CD + AD \cdot BC$

- It holds for a cyclic quadrilateral $ABCD$. For any four points, $AC \cdot BD < AB \cdot CD + AD \cdot BC$, with equality exactly when A, B, C, D lie in this order on a circle, or on a line.
- Van Schooten**: if P is on the arc BC not containing A of the circumcircle of an equilateral triangle ABC , then $PA = PB + PC$.

A regular pentagon with side 1 and diagonal d : Ptolemy on four vertices gives $d^2 = d + 1$, so $d = \frac{1+\sqrt{5}}{2}$.

24 THE CIRCUMCIRCLE C-24

KEY

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, $R = \frac{abc}{4\Delta}$

- The circumcentre O is where the perpendicular bisectors meet. It lies inside, on, or outside the triangle as the triangle is acute, right, or obtuse.
- $\angle BOC = 2A$ when A is acute; when A is obtuse the reflex angle BOC is $2A$, so $\angle BOC = 360^\circ - 2A$.
- For a right triangle, R is half the hypotenuse.

For 13, 14, 15: $R = \frac{2730}{336} = \frac{65}{8}$.

25 INCIRCLE AND EXCIRCLES C-25

tangents from a vertex are equal: $x = a - b$

KEY

$r = \frac{\Delta}{s}$, $r_a = \frac{\Delta}{s-a}$

- The tangent lengths from A, B, C to the incircle are $s - a, s - b, s - c$.
- The excircle opposite A touches BC at a point at distance $s - c$ from B and $s - b$ from C , and touches the extensions of AB and AC at distance s from A .
- In a right triangle with legs a, b and hypotenuse c , $r = \frac{a+b-c}{2}$.

26 TRIANGLE CENTRES C-26

circumcentre, centroid, orthocentre lie on the Euler line, with $OG : GH = 1 : 2$

CENTRE	MEETING OF	FACT
centroid G	medians	divides each 2 : 1
circumcentre O	perpendicular bisectors	$OA = OB = OC = R$
incentre I	angle bisectors	distance r to each side
orthocentre H	altitudes	$AH = 2R \cos A$

KEY

$OH^2 = 9R^2 - (a^2 + b^2 + c^2)$, $OP^2 = R^2 - 2Rr$

- Euler line**: O, G, H are collinear with $OG : GH = 1 : 2$. From $OP^2 \geq 0$, $R \geq 2r$, with equality only for the equilateral triangle.
- The reflection of H in any side lies on the circumcircle.

27 COORDINATES & TECHNIQUES 27-33

27 COORDINATES 0-27

reflect P in z : $PX + XQ = P'X + XQ \geq P'Q$, with equality on the straight line

$PX + XQ = P'X + XQ \geq P'Q$

KEY

$[P_1P_2P_3] = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$

- Put a right angle at the origin with its arms on the axes, or put an axis of symmetry along an axis. Good placement halves the algebra.
- Perpendicular lines have slopes with product -1 ; the distance from the origin to $\frac{x}{p} + \frac{y}{q} = 1$ is $\frac{|pq|}{\sqrt{p^2 + q^2}}$.

28 POLYGONS 0-28

KEY

interior angle of a regular n -gon $= \frac{(n-2) \cdot 180^\circ}{n}$
exterior angle $= \frac{360^\circ}{n}$

- A convex n -gon has $\frac{n(n-3)}{2}$ diagonals.
- A regular n -gon with circumradius R has side $2R \sin \frac{180^\circ}{n}$ and area $\frac{n}{2} R^2 \sin \frac{360^\circ}{n}$.
- A regular hexagon of side s is six equilateral triangles: area $\frac{3\sqrt{3}}{2}s^2$, long diagonal $2s$, short diagonal $\sqrt{3}s$.

Each interior angle 150° means each exterior angle is 30° , so $n = 360/24 = 15$.

29 QUADRILATERALS 0-29

$AB = 3CD$, $CD = DA$; the angles are $60, 30, 150, 120$ degrees

SHAPE	KEY PROPERTY
parallelogram	diagonals bisect each other
rectangle	diagonals equal and bisect each other
rhombus	diagonals perpendicular bisectors of each other
kite	one diagonal is the perpendicular bisector of the other
trapezium	midline $= \frac{1}{2}(a+b)$, area $\frac{1}{2}(a+b)h$
any, diagonals p, q	area $\frac{1}{2}pq \sin \theta$, θ the angle between them

KEY

$[ABCD] = \frac{1}{2}pq \sin \theta$ (not self crossing)

IOQM 2026, Q8: $CD = DA$ with $\angle D = 120^\circ$ gives $AC = \sqrt{3}CD$; then ABC is isosceles with $AC = BC$, and the angles are $60, 30, 150, 120$ degrees, so $x/y = 5$.

30 ANGLE CHASING 0-30

$\angle A = 90^\circ$, $\angle B = 70^\circ$; the sine rule in three triangles gives $s = 40^\circ$

KEY

1. Write every known angle on the figure.
11. Harvest equal angles from parallels, isosceles triangles and cyclic quadrilaterals.
111. If it stalls, call one angle x and express the rest in terms of it.
1111. If it still stalls, use the sine rule in two triangles that share a side, or trigonometric Ceva.

IOQM 2026, Q30: with $\angle A = 90^\circ$, $\angle B = 70^\circ$ and two 30° cevians, the sine rule gives $BE/BD = 2 \sin 50^\circ$, and then $\sin(180^\circ - x) = 2 \sin 50^\circ$ forces $x = 40^\circ$.

Check a guessed angle by substitution. In Q30, $x = 40^\circ$ gives $\frac{\sin 100^\circ}{\sin 60^\circ} = 2 \cos 40^\circ = 2 \sin 50^\circ$, and since the left side decreases in x , the answer is unique.

31 AUXILIARY CONSTRUCTIONS 0-31

- Double a median: extending AM to $2AM$ completes a parallelogram.
- Drop a perpendicular to create right triangles; join the centre to a point of contact to get a right angle.
- Extend the legs of a trapezium to meet, turning it into two similar triangles.
- Extend an angle bisector, or rotate by 60° when equilateral triangles appear.
- Complete a right triangle to a rectangle, or an isosceles triangle to a rhombus.

A good auxiliary line creates a triangle you recognise: isosceles, right, equilateral, or similar to one already in the figure. If a new line makes none of these, it is probably the wrong line.

32 REFLECTION AND EXTREMES 0-32

reflect P in z : $PX + XQ = P'X + XQ \geq P'Q$, with equality on the straight line

$PX + XQ = P'X + XQ \geq P'Q$

KEY

- Shortest path touching a line**, for P and Q on the same side of it: reflect one end in the line and join straight; the path makes equal angles with the line.
- Among triangles with a fixed perimeter, and among rectangles with a fixed perimeter, the equilateral triangle and the square have the largest areas.
- Fermat point**: when every angle of the triangle is below 130° , the point minimising $PA + PB + PC$ sees each side at 120° .

33 LATTICE POINTS AND PICK'S THEOREM 0-33

$P = 11$ inside
 $B = 4$ on the edge
area $12 = 11 + \frac{1}{2} \cdot 4$

$A = I + \frac{B}{2} - 1$

KEY

- For a polygon whose vertices are lattice points, I counts the lattice points inside and B those on the boundary.
- The segment from (x_1, y_1) to (x_2, y_2) passes through $\gcd(|x_2 - x_1|, |y_2 - y_1|) + 1$ lattice points, ends included.
- A lattice triangle has area at least $\frac{1}{2}$; and no equilateral triangle has all three vertices at lattice points, since its area $\frac{\sqrt{3}}{4}s^2$, with s^2 a positive integer, is irrational, while a lattice triangle's area is a multiple of $\frac{1}{2}$.

34 METHOD & TRAPS 34-38

34 WHICH TOOL THE QUESTION WANTS C-34

What does the question want?

- an angle: chase it first. Parallels, isosceles triangles and cyclic quadrilaterals supply most of the equalities.
- a length or ratio: find a pair of similar triangles, or use the bisector theorem, Stewart, or the cosine rule.
- an area: compare triangles with a common height, use $\frac{1}{2}ab \sin C$, or put the figure on coordinates.
- any circle in the figure: inscribed angles, tangents and power of a point come before anything else.

35 STANDARD VALUES WORTH CARRYING C-35

FIGURE	VALUES
equilateral, side a	$h = \frac{\sqrt{3}}{2}a$, $\Delta = \frac{\sqrt{3}}{4}a^2$, $R = 2r = \frac{a}{\sqrt{3}}$
3, 4, 5 triangle	$\Delta = 6$, $r = 1$, $R = \frac{5}{2}$
13, 14, 15 triangle	$\Delta = 84$, $r = 4$, $R = \frac{65}{8}$, height to 14 is 12
square, side a	diagonal $\sqrt{2}a$, $R = \frac{a}{\sqrt{2}}$, $r = \frac{a}{2}$
regular hexagon, side a	$\Delta = \frac{3\sqrt{3}}{2}a^2$, $R = a$, $r = \frac{\sqrt{3}}{2}a$
regular pentagon, side l	diagonal $\frac{l\sqrt{5}}{2}$

36 HOW IOQM 2026 USED THIS SHEET E-36

Q	ASKED FOR	TOOL, SECTION
5	an angle at an arc midpoint	arc midpoints, 18
8		