

CONCEPT MAP

IOQM · ALGEBRA

5 TOPICS · 38 CORE IDEAS · ONE SHEET



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A IDENTITIES & MANIPULATION 01-08

01 IDENTITIES WORTH KNOWING A-01

- $(a \pm b)^2, (a \pm b)^3, a^2 - b^2, a^3 \pm b^3$, and $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$ should be automatic.

KEY

$$a^3 + b^3 + c^3 - 3abc \\ = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

- So $a + b + c = 0$ gives $a^3 + b^3 + c^3 = 3abc$.
- The second factor is $\frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$, which for reals is never negative and is zero only when $a = b = c$.

The three terms $x - y, y - z, z - x$ add to 0, so $(x - y)^3 + (y - z)^3 + (z - x)^3 = 3(x - y)(y - z)(z - x)$.

02 FACTORISATION TECHNIQUES A-02

- Take out common factors and group terms.
- **Complete a square, then use a difference of squares:** $x^4 + x^2 + 1 = (x^2 + 1)^2 - x^2$.
- Read the expression as a quadratic in one variable; it factors over the rationals when its discriminant is the square of a polynomial with rational coefficients.
- **Pair the factors** of a product of four linear terms so that the two quadratics share $x^2 + kx$.

KEY

$$x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1)$$

$(x+1)(x+4) \cdot (x+2)(x+3) + 1 = (u+4)(u+6) + 1 = (u+5)^2$ with $u = x^2 + 5x$, so it equals $(x^2 + 5x + 5)^2$.

03 SYMMETRIC FUNCTIONS OF TWO VARIABLES A-03

Every symmetric polynomial in x, y is a polynomial in $s = x + y$ and $p = xy$.

KEY

$$x^n + y^n = s(x^{n-1} + y^{n-1}) - p(x^{n-2} + y^{n-2})$$

- $x^2 + y^2 = s^2 - 2p, x^3 + y^3 = s^3 - 3sp, (x - y)^2 = s^2 - 4p$.
 - x and y are the roots of $t^2 - st + p = 0$, so they are real exactly when $s^2 \geq 4p$.
- $x + y = 3, xy = 1: x^3 + y^3 = 27 - 9 = 18$ and $x^4 + y^4 = (9 - 2)^2 - 2 = 47$.

04 SURDS AND NESTED RADICALS A-04

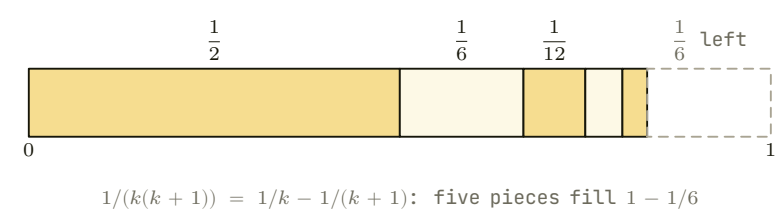
KEY

$$\sqrt{a \pm 2\sqrt{b}} = \sqrt{m} \pm \sqrt{n} \\ (m + n = a, mn = b, m, n \geq 0)$$

- Rationalise with the conjugate: $\frac{1}{\sqrt{a} + \sqrt{b}} = \frac{\sqrt{a} - \sqrt{b}}{a - b}$ for $a \neq b$.
 - So $\sum_{k=1}^n \frac{1}{\sqrt{k} + \sqrt{k+1}} = \sqrt{n+1} - 1$, a telescoping sum.
 - If $a + b\sqrt{2} = c + d\sqrt{2}$ with a, b, c, d rational, then $a = c$ and $b = d$, because $\sqrt{2}$ is irrational.
- $7 + 4\sqrt{3} = 7 + 2\sqrt{12}$, and $4 + 3 = 7, 4 \cdot 3 = 12$, so $\sqrt{7 + 4\sqrt{3}} = 2 + \sqrt{3}$.

05 TELESCOPING A-05

KEY

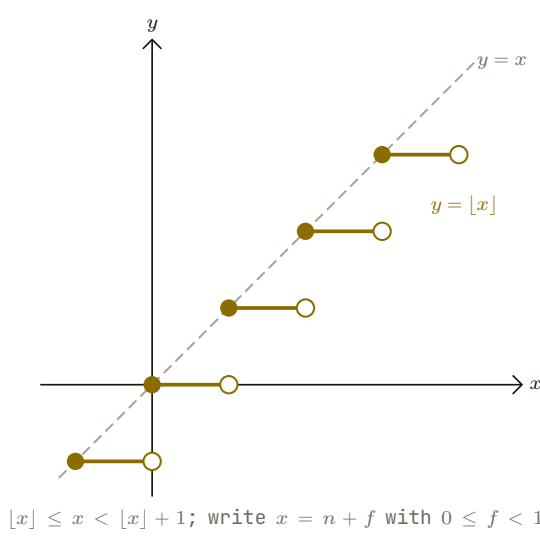


$$\sum_{k=1}^n (f(k) - f(k+1)) = f(1) - f(n+1)$$

- $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$, and $\frac{1}{k(k+1)(k+2)} = \frac{1}{2} \left(\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)} \right)$.
 - $k \cdot k! = (k+1)! - k!$.
 - Products telescope too: $\prod_{k=2}^n \left(1 - \frac{1}{k^2} \right) = \prod_{k=2}^n \frac{(k-1)(k+1)}{k^2} = \frac{n+1}{2n}$.
- $\sum_{k=1}^{99} \frac{1}{k(k+1)} = 1 - \frac{1}{100} = \frac{99}{100}$

06 FLOOR AND FRACTIONAL PART A-06

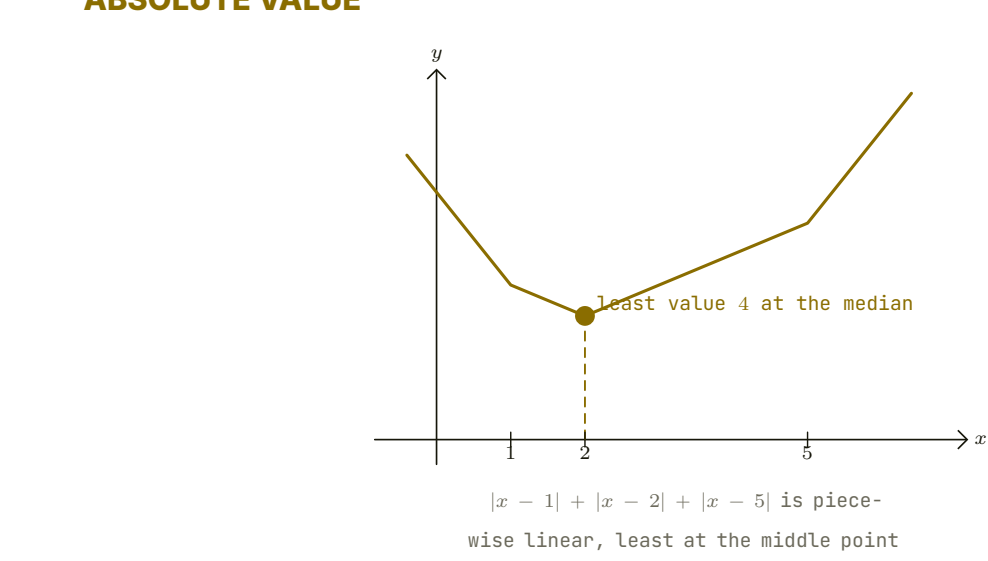
KEY



$$x = [x] + \{x\}, \quad 0 \leq \{x\} < 1$$

- $[x + n] = [x] + n$ for every integer n , and $[x] + [y] \leq [x + y] \leq [x] + [y] + 1$.
 - $[-x] = -[x] - 1$ when x is not an integer.
 - **Hermite:** $\sum_{k=0}^{n-1} [x + \frac{k}{n}] = [nx]$ for every positive integer n .
 - **To solve:** write $x = n + f$ with n an integer and $0 \leq f < 1$, and turn the equation into conditions on n and f .
- $x + [x] = 8.4: 2n + f = 8.4$ with $0 \leq f < 1$ forces $n = 4, f = 0.4$, so $x = 4.4$.

07 ABSOLUTE VALUE A-07



KEY

$$|a + b| \leq |a| + |b|, \quad ||a| - |b|| \leq |a - b|$$

- Equality in $|a + b| \leq |a| + |b|$ holds exactly when $ab \geq 0$.
- To solve an equation in absolute values, split the line at the zeros of the expressions inside.
- $\sum |x - a_i|$ is least at a median of the a_i for an even number of points, anywhere between the two middle ones.

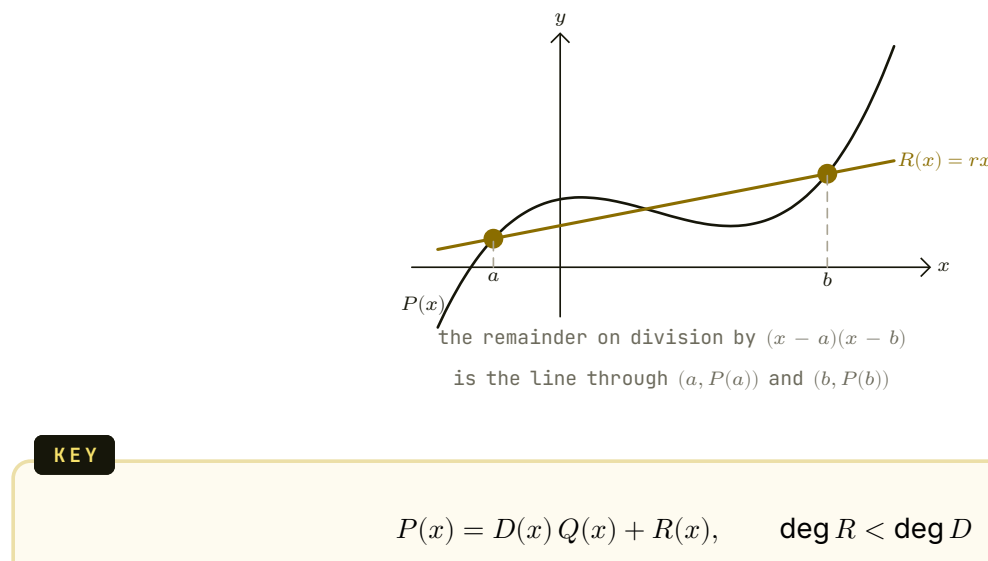
08 COMPARING LARGE NUMBERS A-08

- **Match the exponents:** $2^{300} = 8^{100} < 9^{100} = 3^{200}$.
- **Insert a convenient middle term:** $3^{11} < 32^{11} = 2^{55} < 2^{56} = 16^{14} < 17^{14}$.
- For $n \geq 3$, $n^{n+1} > (n+1)^n$; so $99^{100} > 100^{99}$.
- Compare a ratio with 1 when the two numbers are close.

Never compare by estimating digits alone. The safe method is a chain of inequalities that are each obvious, as in the second example.

B POLYNOMIALS 09-17

09 REMAINDER AND FACTOR THEOREMS 0-09



KEY

$$P(x) = D(x)Q(x) + R(x), \quad \deg R < \deg D$$

- The remainder on division by $x - a$ is $P(a)$, so $x - a$ is a factor exactly when $P(a) = 0$.
- On division by $(x - a)Q(x)$ with $a \neq b$, the remainder has degree at most 1, say $rx + s$, fixed by $P(a)$ and $P(b)$.

$P(1) = 3$ and $P(2) = 5$: the remainder on division by $(x-1)(x-2)$ is $2x + 1$.

10 VIETA'S FORMULAS 0-10

If $a_n x^n + \dots + a_0$ has roots $\alpha_1, \dots, \alpha_n$, the k th elementary symmetric sum e_k of the roots is

KEY

$$e_k = (-1)^k \frac{a_{n-k}}{a_n}$$

- Quadratic $ax^2 + bx + c$: $\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$.
 - Cubic $ax^3 + bx^2 + cx + d$: $\sum \alpha = -\frac{b}{a}, \sum \alpha\beta = \frac{c}{a}, \alpha\beta\gamma = -\frac{d}{a}$.
- $x^3 - 6x^2 + 11x - 6$ has roots 1, 2, 3: sum 6, pair sum 11, product 6.

Vieta answers symmetric questions without finding the roots. Any symmetric expression in the roots is a polynomial in the e_k , and so in the coefficients.

11 POWER SUMS AND NEWTON'S IDENTITIES 0-11

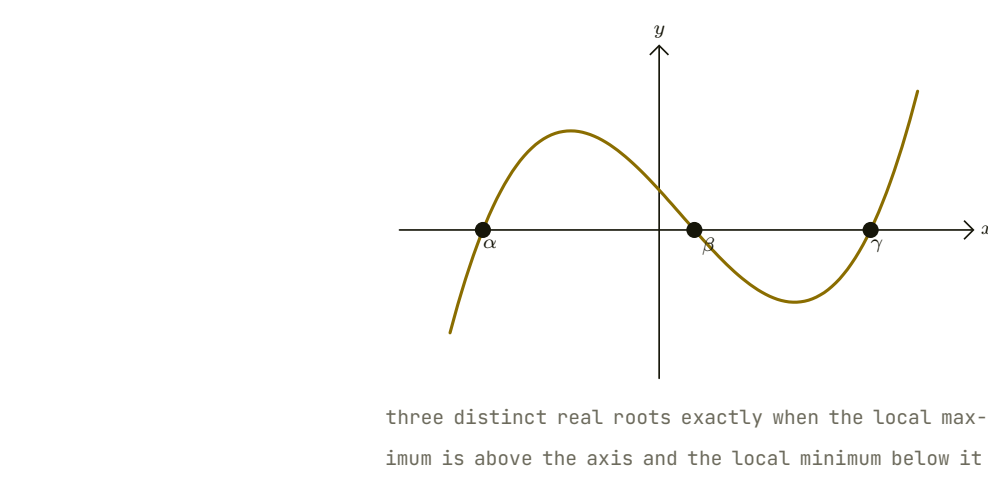
With $p_k = \sum \alpha_i^k$,

KEY

$$p_1 = e_1, \quad p_2 = e_1 p_1 - 2e_2 \\ p_3 = e_1 p_2 - e_2 p_1 + 3e_3$$

- For the roots of $x^2 - sx + q$, $p_k = s p_{k-1} - q p_{k-2}$ for $k \geq 2$, with $p_0 = 2$.
 - $\sum \frac{1}{\alpha_i} = \frac{e_{n-1}}{e_n}$ when no root is zero.
- Roots of $x^3 - x - 1$: $e_1 = 0, e_2 = -1, e_3 = 1$, so $p_2 = 2$ and $p_3 = 3$; check: each root has $\alpha^3 = \alpha + 1$, so $p_3 = p_2 + 3 = 5$.

12 ROOTS AND GRAPHS 0-12



- Real roots are where the graph meets the x -axis. If a polynomial takes opposite signs at a and b , it has a root between them.
- A real polynomial of odd degree has at least one real root, and its non real roots come in conjugate pairs.
- A polynomial of degree $n \geq 1$ has exactly n complex roots counted with multiplicity, so at most n real ones.
- At a root of even multiplicity the graph touches the axis without crossing it; at a root of odd multiplicity it crosses.
- **Descartes:** the number of positive roots, counted with multiplicity, is at most the number of sign changes in the sequence of non zero coefficients, and differs from it by an even number.

13 INTEGER AND RATIONAL ROOTS 0-13

- **Rational root theorem:** If p/q in lowest terms is a root of a polynomial with integer coefficients, then $p \mid a_0$ and $q \mid a_n$.
- For a monic integer polynomial, every rational root is an integer dividing a_0 .

KEY

$$a - b \mid P(a) - P(b) \quad (\text{integer coefficients and } a, b)$$

- If $P(0)$ and $P(1)$ are both odd, P has no integer root, since $P(n)$ always has the parity of $P(0)$ or $P(1)$.
- $2x^3 - 3x^2 - 3x + 2$: test $\pm 1, \pm 2, \pm \frac{1}{2}$; the roots are 2, -1 and $\frac{1}{2}$.

14 COMPARING POLYNOMIALS 0-14

KEY

$$\deg P(Q(x)) = \deg P \cdot \deg Q \\ \deg P'Q = \deg P + \deg Q \quad (P, Q \neq 0)$$

- For the first formula Q must also be non constant. Two polynomials of degree at most n that agree at $n+1$ points are identical.
- Compare degrees first, then leading coefficients, then the rest.
- Special values, $x = 0, \pm 1$ or known roots, give linear equations in the coefficients.

IOQM 2026, Q11: $P(x^2) = P(P(x))$ with $\deg P = n \geq 1$ gives $2n = n^2$, so $n = 2$; comparing coefficients leaves $P(x) = x^2$ and $P(x) = -x^2$, two polynomials.

15 INTERPOLATION AND FINITE DIFFERENCES 0-15

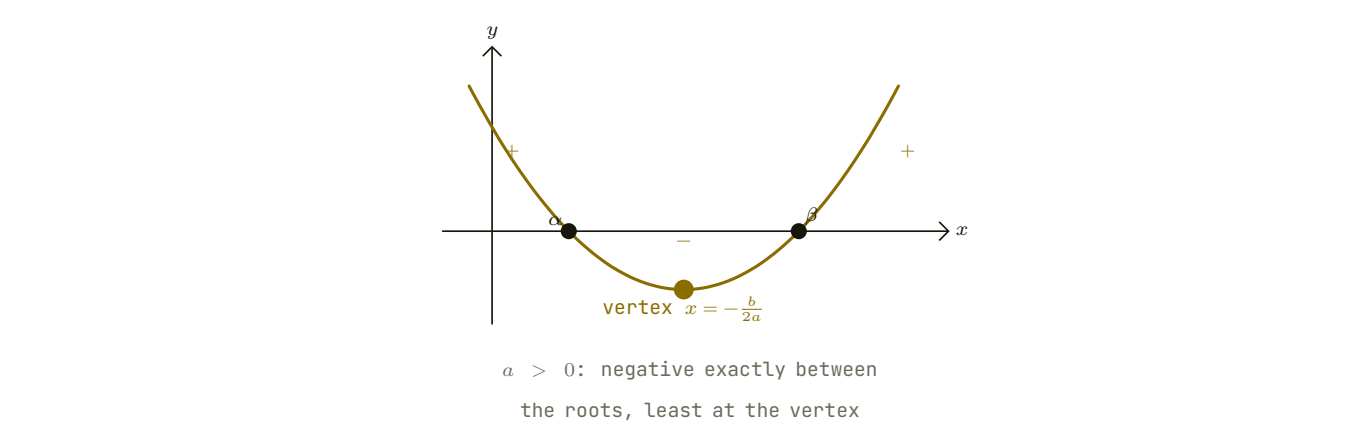
- Through $n+1$ points with distinct x values there is exactly one polynomial of degree at most n .
- **Build a helper with known roots.** If $\deg P = n-1$ and $P(k) = \frac{1}{k}$ for $k = 1, \dots, n$, then $Q(x) = xP(x) - 1$ vanishes at $1, \dots, n$, so $Q(x) = c(x-1)\dots(x-n)$, and $Q(0) = -1$ fixes c .
- The n th difference of a polynomial of degree n is constant:

KEY

$$\Delta^n P = n! a_n$$

A cubic with $P(1), \dots, P(4) = 1, 2, 4, 8$; differences 1, 2, 4, then 1, 2, then 1. Keeping the third difference at 1 gives $P(5) = 15$.

16 QUADRATICS 0-16

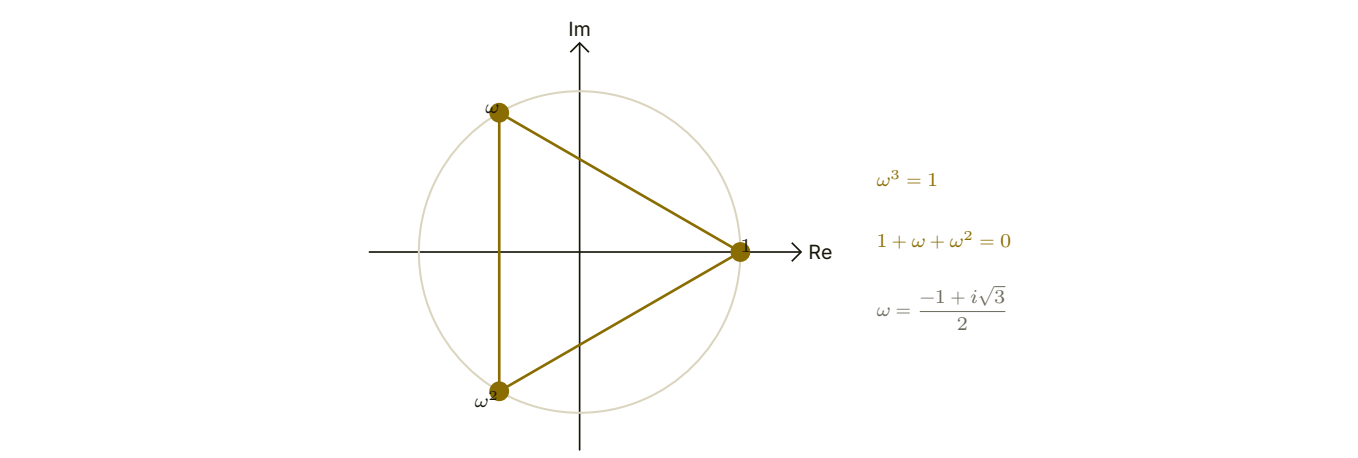


KEY

$$ax^2 + bx + c = a \left(x + \frac{b}{2a} \right)^2 - \frac{b^2 - 4ac}{4a}$$

- With $D = b^2 - 4ac$: two real roots for $D > 0$, a double root for $D = 0$, none for $D < 0$. With integer coefficients, the roots are rational exactly when D is a perfect square.
- For $a > 0$ the least value is $-\frac{D}{4a}$ at $x = -\frac{b}{2a}$.
- Both roots exceed k exactly when $D \geq 0, a f(k) > 0$ and $-\frac{b}{2a} > k$.

17 COMPLEX NUMBERS AND ROOTS OF UNITY 0-17



KEY

$$x^n - 1 = \prod_{k=0}^{n-1} (x - \omega^k), \quad \omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$$

- For $n \geq 2$ the n th roots of unity add to 0.
- For cube roots: $\omega^3 = 1, 1 + \omega + \omega^2 = 0, \bar{\omega} = \omega^2$, and $x^2 + x + 1 = (x - \omega)(x - \omega^2)$.
- **Roots of unity filter:** $\sum_{j \equiv k \pmod n} \binom{n}{j} = \frac{1}{n} (2^n + (1 + \omega)^n + (1 + \omega^2)^n) = \frac{1}{n} (2^n + 2 \cos \frac{2\pi k}{n})$.

C EQUATIONS & SYSTEMS 18-25

18 SYMMETRIC AND CYCLIC SYSTEMS 0-18

- Subtract two equations to pull out a factor such as $x - y$; then split into $x = y$ or the other factor = 0.
- Add all the equations, or multiply them, when the system is cyclic.
- For a symmetric system in two unknowns, pass to $s = x + y$ and $p = xy$.

IOQM 2026, Q21: subtracting gives $(x-z)(x+z-1-y) = 0$ and its cyclic copies. All distinct is impossible; all equal gives $t = 0$ or -1 ; exactly two equal gives $x = y = \pm \frac{1}{2}$, $z = 1$ and its cyclic copies. The values are 0, 8, 4, total 12.

A case split must be exhaustive. For three unknowns that means all equal, exactly two equal, and all distinct; missing one case loses solutions, as the complex ones in Q21 show.

19 SUBSTITUTIONS 0-19

KEY

$$x^2 + \frac{1}{x^2} = t^2 - 2, \quad x^3 + \frac{1}{x^3} = t^3 - 3t$$

- Here $t = x + \frac{1}{x}$. **Reciprocal equations** of even degree n , whose coefficients read the same both ways: divide by $x^{n/2}$ and set $t = x + \frac{1}{x}$. One of odd degree has the factor $x + 1$; remove it first.
 - Repeated blocks: set u equal to the repeated expression, as $u = x^2 + 5x$ in section 2.
 - Homogeneous equations in x, y : divide by a power of y and set $t = x/y$.
- $x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$: dividing by x^2 gives $t^2 - 3t + 2 = 0$, so $t = 1$ or 2. Only $t = 2$ gives real x , namely $x = 1$, a double root.

20 INTEGER OPTIMISATION AND PARITY 0-20

KEY

$$x^2 \equiv x \pmod{2} \implies \sum x_i^2 \equiv \sum x_i \pmod{2}$$

- A non zero integer has square at least 1, and no integer square equals 2, 3, 5, 6, 7 or 9.
- For integers with a fixed sum, $\sum x_i^2$ is least when they are as equal as possible, differing by at most 1.

IOQM 2026, Q3: 49 non zero integers with sum 0 have $\sum x_i^2 \geq 49$ and even, so at least 50. Fifty would need one square equal to 2, so the minimum is 52, with minus one and one 2.

21 FUNCTIONAL EQUATIONS: FIRST MOVES 0-21

- Substitute $x = 0, y = 0, y = x, y = -x, y = 1$ to get values of f and relations.
- **Prove f** is one to one or onto, then use it to cancel or to reach every value.
- **Cauchy:** $f(x+y) = f(x) + f(y)$ gives $f(q) = q f(1)$ for every rational q ; with continuity, monotonicity or boundedness on an interval, $f(x) = cx$ for all reals.

KEY

$$f(x) + 2f\left(\frac{1}{x}\right) = 3x \\ \xrightarrow{x \rightarrow 1/x} f\left(\frac{1}{x}\right) + 2f(x) = \frac{3}{x}$$

Solving the pair above as two linear equations gives $f(x) = \frac{2}{x}$ for $x \neq 0$, which does satisfy the original.

Always verify the answer in the original equation. Substitutions produce necessary conditions only; a function found this way still has to be checked.

22 POLYNOMIAL FUNCTIONAL EQUATIONS 0-22

- Compare degrees, then leading coefficients, then the remaining coefficients, as in IOQM 2026, Q11.
- The roots of $P(x) - x$ are the fixed points of P .
- If $P(x+1) - P(x)$ is known, P is fixed up to an added constant.

$P(x)^2 = P(x^2)$: the leading coefficient a has $a^2 = a$, so $a = 1$; the non constant real solutions are exactly $P(x) = x^n$.

23 RADICALS AND EXTRANEEOUS ROOTS 0-23

- State the domain first: expressions under even roots are non negative, denominators are non zero.
- Squaring both sides can create roots that do not solve the original; test every candidate.

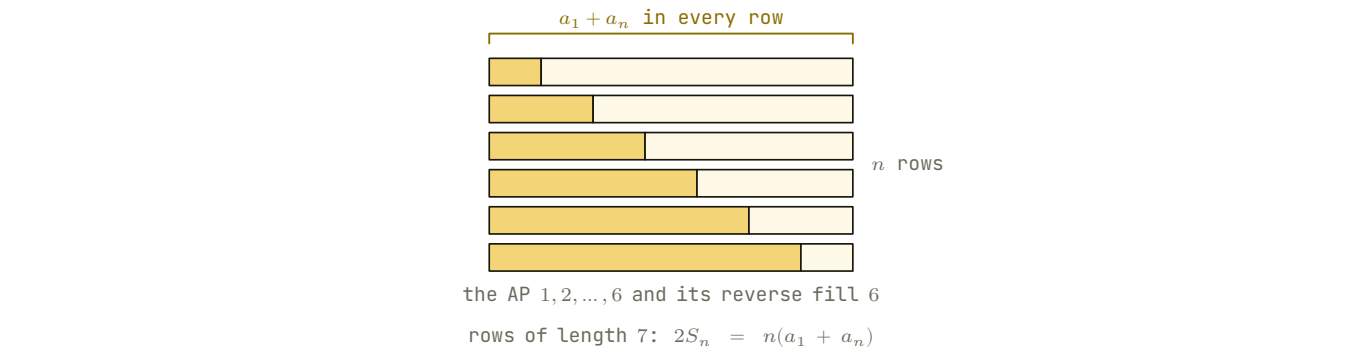
KEY

$$\sqrt{A} = B \iff A = B^2 \text{ and } B \geq 0$$

$\sqrt{x+5} = x-1$: squaring gives $x^2 - 3x - 4 = 0$, so $x = 4$ or -1 ; the condition $x-1 \geq 0$ rejects -1 , leaving $x = 4$.

D SEQUENCES & INEQUALITIES 24-33

24 PROGRESSIONS 0-24



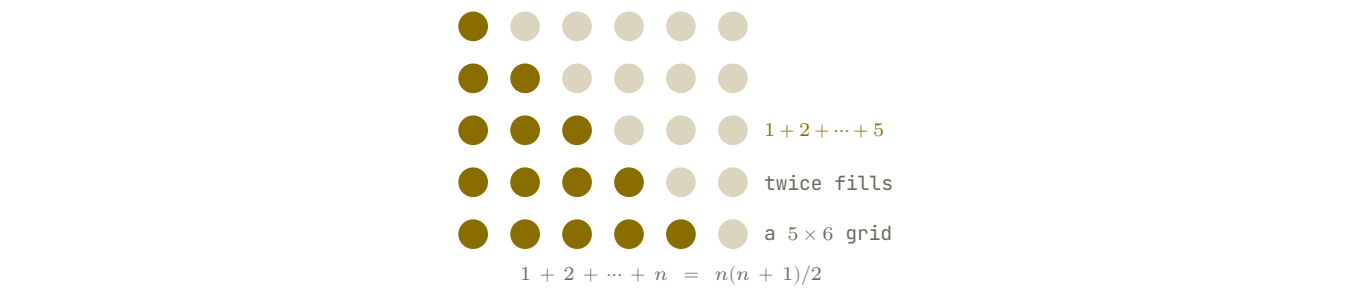
KEY

$$S_n = \frac{n}{2} (a_1 + a_n), \quad S_n = a \frac{r^n - 1}{r - 1} \quad (r \neq 1)$$

- An infinite GP with $|r| < 1$ sums to $\frac{a}{1-r}$.
- a, b, c are in AP when $2b = a + c$; in GP when $b^2 = ac$ with a, b, c non zero, in HP when $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$.
- **Arithmetic geometric:** multiply the sum by r and subtract, leaving a plain GP.

$$\sum_{k=1}^n \frac{1}{2^k} = 2.$$

25 SUMS OF POWERS 0-25

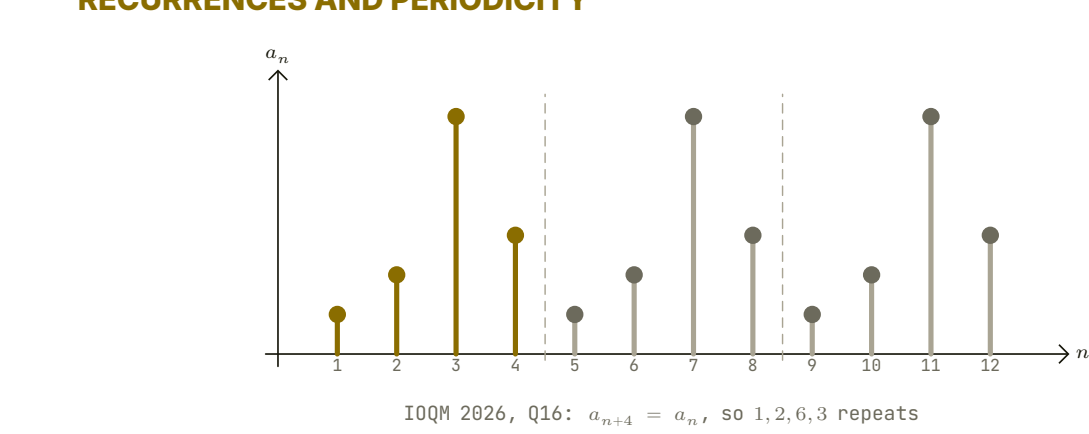


KEY

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6} \\ \sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2} \right)^2$$

- $\sum_{k=1}^n k(k+1) \dots (k+r-1) = \frac{n(n+1) \dots (n+r)}{r+1}$, a telescoping sum.
- The first n odd numbers add to n^2 , and $\sum_{k=1}^n k \cdot k! = (n+1)! - 1$.

26 RECURRENCES AND PERIODICITY 0-26



- Compute the first few terms: many olympiad sequences repeat.
- $x_{n+1} = \frac{1}{1-x_n}$ has period 3 when $x_1 \neq 0, 1$: $x \mapsto \frac{1}{1-x} \mapsto \frac{x-1}{x} \mapsto x$.
- $x_{n+1} = \frac{1+x_n}{1-x_n}$ has period 4 when $x_1 \neq 0, \pm 1$, and $a_{n+1} = a_n - a_{n-1}$ has period 6.

IOQM 2026, Q16: cross multiplying gives $a_{n+3}a_{n+1} = a_{n+2}a_n$, so $a_{n+1}a_{n+2} = a_{n+2}a_n$, and the terms being non zero, $a_{n+1} = a_n$. The block 1, 2, 6, 3 gives $N = 506 \cdot 50 + 5 = 25305$, digit sum 15.

Cancelling a term needs it to be non zero. State it; in Q16 it follows because the given ratios have non zero denominators and the known terms are non zero.

27 LINEAR RECURRENCES AND FIBONACCI 0-27

- $a_n = p a_{n-1} + q a_{n-2}$ is solved through the roots of $x^2 = px + q$, as on the IOQM Combinatorics sheet.
- **Binet:** $F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$ with $\phi = \frac{1+\sqrt{5}}{2}, \psi = \frac{1-\sqrt{5}}{2}$.

KEY

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

- $F_1 + \dots + F_n = F_{n+2} - 1$ and $F_1^$