

[OLYMPIAD MATHEMATICS]

IOQM · PRE-RMO

2027

IOQM *Handbook*

CLASS 8 TO 12

OLYMPIAD

THEORY AND PROBLEMS

IN THIS HANDBOOK

PAPER	IOQM MATHEMATICS
THIS VOLUME	PART II · COMBINATORICS
EACH CHAPTER	THEORY · EXAMPLES · EXERCISES

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IOQM Mathematics Handbook

Part II · Combinatorics

FOR IOQM 2027 | EDITION 1 | 2026

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Every answer in this handbook has been checked by computer. If you still find an error, or if you have a suggestion, please write to ritesh@iitianforum.com or academics@iitianforum.com, with the page number. Corrections are made in the next edition.

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[IOQM · PART II]

Preface

IOQM is the first stage of the Indian mathematical olympiad programme, and for most students it is the first paper in which a formula is not enough. The questions are short and every answer is a whole number from 00 to 99, but almost every question needs an idea. This handbook is written to build those ideas, one chapter at a time.

The handbook follows the paper. It has four parts, one each for number theory, combinatorics, geometry and algebra, and every part is tied to the IOQM 2026 paper, so that you can see where each idea was actually tested. This volume is Part II, Combinatorics. Nine questions of IOQM 2026 were combinatorics questions, second only to number theory, and many of them were half a count and half an argument.

Everything here stays within school mathematics. Nothing needs more than the permutations and combinations of Classes 11 and 12, and every result that is used is either proved in the text or stated with its conditions. The examples and exercises come from the IOQM 2026 paper, from olympiads of many countries, and from our own classes. Every solution has been written afresh, in the way we teach it, and every answer has been checked by computer.

This book is meant to be worked through, not only read. Keep a notebook beside it, attempt every example before you read its solution, and come back later to the questions you could not do.

If something is not clear, if you find a mistake, or if you have a suggestion, write to me at ritesh@iitianforum.com or to our academic team at academics@iitianforum.com. Your message will help every student who uses this book after you.

Ritesh Raj

Founder, IITian Forum · IIT Delhi

[IOQM · PART II]

How to use this handbook

This page is for you, the student. Part II has five chapters, in the order in which they are taught. Chapter 1 covers the basic principles of counting. Chapter 2 is about distributions, grid paths and bijections, and Chapter 3 about inclusion and exclusion and recursions. Chapter 4 covers the pigeonhole principle, colouring and invariants. Chapter 5 covers graphs, shows how to choose the right method, and ends with a mixed problem set. The answers to all exercises, each with a one-line hint, are at the end of the part. This part can be read on its own.

HOW TO STUDY A CHAPTER

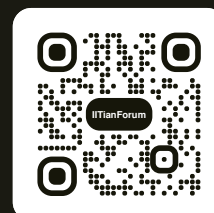
- 01** Read the theory slowly, once. Write out each short proof in your own notebook; that is how a result becomes yours.
- 02** Before reading a worked example, cover its solution and try the question for five minutes. Then read the solution and compare it with yours.
- 03** Learn the gold cards by heart. They hold the definitions, theorems and key ideas that the paper uses again and again.
- 04** Read the dark cards twice. Each one marks a point where students often lose marks.
- 05** Attempt every exercise before you look at the answers. Read the one-line hint only when you are stuck, then finish the question yourself.
- 06** Keep a list of the questions you got wrong, and try them again a week later without looking at the solution.

Questions from IOQM 2026 are marked with their question number, and questions from other olympiads with their source. Stuck on a question, or found a mistake? Write to ritesh@iitianforum.com or academics@iitianforum.com.

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SCAN FOR THE COURSE

[IOQM · PART II]

Notation and conventions

Each symbol is defined, or first used, in the section shown.

SYMBOL	MEANING	SECTION
$\lfloor x \rfloor, \lceil x \rceil$	the greatest integer $\leq x$; the least integer $\geq x$	1.1
$n!$	$1 \cdot 2 \cdot 3 \cdots n$, the number of orders of n different objects; $0! = 1$	1.2
$P(n, r)$	$n! / (n - r)!$, ordered choices of r objects from n	1.2
$\binom{n}{r}$	$n! / (r! (n - r)!)$, choices of r objects from n , order not counted	1.3
$\#(\dots)$	the number of objects of the kind described	1.5
$S(n, k)$	ways to split n different objects into k non-empty groups, not named	2.3
C_n	the n th Catalan number, $\frac{1}{n+1} \binom{2n}{n}$	2.5
$p(n)$	the number of partitions of n	2.8
$ A , A \cup B, A \cap B$	the number of elements of A ; the union and the intersection of A, B	3.1
D_n	the number of derangements of n objects	3.2
$a \bmod m$	the remainder of a on division by m	4.3
$\deg(v), E$	the number of edges at a vertex v ; the number of edges of a graph	5.1
K_n	the graph on n vertices with every pair joined	5.2
$a \mid b, a \nmid b$	a divides b ; a does not divide b (used in the answers)	
$\Rightarrow, \iff$	"implies"; "exactly when" (if and only if)	

CONVENTIONS

- (i) **Different and identical.** People, books and labelled objects are different; sweets, balls and coins of one kind are identical, unless the question says otherwise.
- (ii) **Order.** Arrangements in a row are ordered. A selection, a group or a committee is not.
- (iii) **Circles.** Arrangements round a circle that differ by a rotation are the same. For a necklace, a reflection is also the same.

- (iv) **Numbers.** The first digit of a number is never 0.
- (v) **Least and largest.** A least or largest value needs two things: a bound, and an example that reaches it.
- (vi) **Numbering.** 3.5 is section 5 of Chapter 3. Examples are numbered in the same way, and Exercise 2.5 is question 5 of Exercises 2.
- (vii) **Sources.** "(IOQM 2026, Q17)" is question 17 of the IOQM 2026 paper. The name of another olympiad, as in "(ARML 2004)", gives the source of that question.

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CHAPTER 01 COUNTING PRINCIPLES

A counting question asks “how many”. The answer comes from splitting the objects into cases that do not overlap, or from building each object in stages, and then checking that nothing has been counted twice or left out. This chapter covers the two basic principles, permutations and combinations, circular arrangements, counting the complement, objects kept together or apart, numbers with digit conditions, and how to correct an overcount.

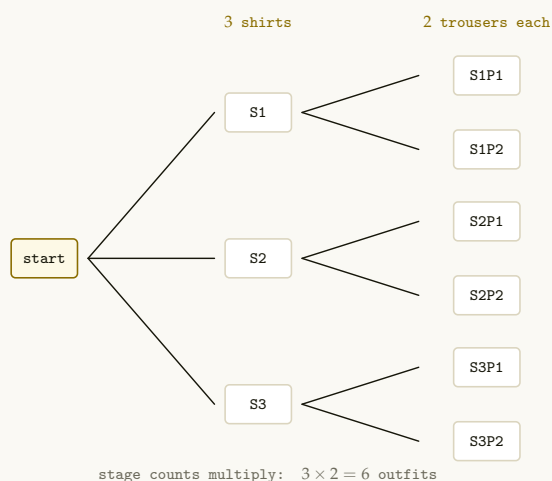
1.1 ADDITION AND MULTIPLICATION

THEOREM · THE TWO PRINCIPLES

Addition: if the objects split into cases that do not overlap, with m in one case and n in the other, there are $m + n$ in all.

Multiplication: if each object is built in stages, and stage i always offers n_i choices, whatever was chosen before, there are $n_1 n_2 \cdots n_k$ objects.

What must stay fixed is the *number* of choices at each stage, not the choices themselves. For example, a divisor of $72 = 2^3 \cdot 3^2$ is built by choosing the power of 2 (4 ways) and then the power of 3 (3 ways), so 72 has 12 divisors.



EXAMPLE 1.1

How many ordered triples (a, b, c) of integers with $1 \leq a \leq b \leq c \leq 40$ satisfy $ab = c$? (CMIMC 2019)

$c = ab$ is fixed by a and b . So we count the pairs (a, b) with

$$a \leq b, \quad ab \leq 40$$

Then $c = ab \geq b$ also holds, as $a \geq 1$.

$a \cdot a \leq a \cdot b \leq 40$ (as $a \leq b$), so $a \leq 6$. For each a , $a \leq b \leq \lfloor 40/a \rfloor$:

a	values of b	count
1	1, ..., 40	40
2	2, ..., 20	19
3	3, ..., 13	11
4	4, ..., 10	7
5	5, ..., 8	4
6	6	1

Total = $40 + 19 + 11 + 7 + 4 + 1 = 82$.

ANSWER 82

1.2 PERMUTATIONS

RULES · ARRANGEMENTS

n distinct objects in a row: $n!$ ways.

An ordered choice of r of them: $P(n, r) = \frac{n!}{(n-r)!}$.

r places, each filled from n options with repetition: n^r .

n objects with a alike of one kind, b alike of another, and so on: $\frac{n!}{a! b! \cdots}$.

For example, BANANA has three A, two N and one B, so it has $\frac{6!}{3! 2!} = 60$ arrangements. We take $0! = 1$; then $P(n, n) = n!$ needs no special case.

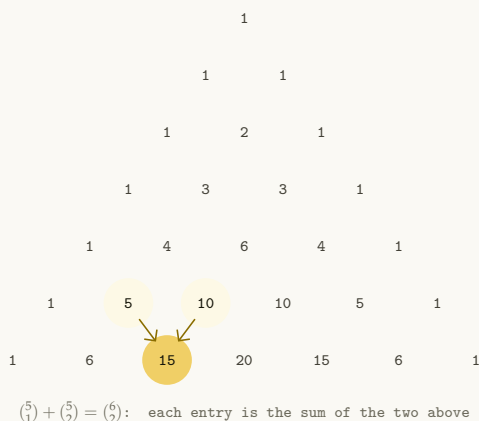
1.3 COMBINATIONS

DEFINITION · COMBINATIONS

The number of ways to choose r objects from n distinct objects, when the order does not matter, is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

- (i) $\binom{n}{r} = \binom{n}{n-r}$: choosing the r to take is choosing the $n-r$ to leave.
- (ii) **Pascal:** $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$, as a chosen group either contains a fixed object or does not.
- (iii) $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$, all subsets.
- (iv) **Hockey stick:** $\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$.



EXAMPLE 1.2

Each square of a 3×3 grid is coloured red or green. In how many colourings does no row and no column contain more than one green square? (CMIMC 2017)

No two green squares share a row or a column. There are 3 rows, so there are 0, 1, 2 or 3 green squares.

0 green: 1 way. 1 green: 9 ways (any square).

2 green: their 2 rows in 3 ways, their 2 columns in 3 ways. The upper square takes one of these columns and the lower one the other: 2 ways.

$$3 \cdot 3 \cdot 2 = 18$$

3 green: one in each row and each column. Row 1: 3 columns, row 2: 2 left, row 3: 1, so $3 \cdot 2 \cdot 1 = 6$. On adding,

$$1 + 9 + 18 + 6 = 34$$

ANSWER 34

1.4 CIRCULAR ARRANGEMENTS

- (i) n distinct objects round a circle, rotations counted as the same: $(n-1)!$. Fix one object and arrange the rest in a line.
- (ii) If turning over is also allowed, as for a necklace with $n \geq 3$ beads: $\frac{(n-1)!}{2}$.
- (iii) If the seats are numbered, it is a line again: $n!$.

For example, 5 boys and 5 girls sit alternately at a round table: seat the boys in $4!$ ways, then the girls in the 5 gaps in $5!$ ways, 2880 in all.

EXAMPLE 1.3

In how many ways can 6 people sit at a round table if two of them, A and B, must not sit together?

All seatings of 6 people:

$$(6-1)! = 120$$

A, B together: tie them into one block. The 5 units are arranged round the table in $(5-1)! = 24$ ways, and the block is AB or BA: 2 ways.

$$\text{together : } 24 \cdot 2 = 48$$

$$\text{not together : } 120 - 48 = 72$$

ANSWER 72

WHERE MARKS GO · DIVIDING BY n

$(n-1)!$ comes from dividing $n!$ by the n rotations, which needs distinct objects. Two red and two blue beads give 6 rows but only 2 circles up to rotation, not $6/4$.

1.5 COUNTING THE COMPLEMENT

KEY IDEA · COUNT WHAT YOU DO NOT WANT

$$\#(\text{at least one}) = \#(\text{all}) - \#(\text{none}).$$

Use it when the unwanted objects form one simple case and the wanted ones split into many.

EXAMPLE 1.4

How many 4-digit numbers contain at least one digit 7?

Four-digit numbers: 1000 to 9999, i.e., 9000 numbers. Those with no 7 are easier to count.

First digit: not 0, not 7, so 8 choices. Each other digit: not 7, so 9 choices.

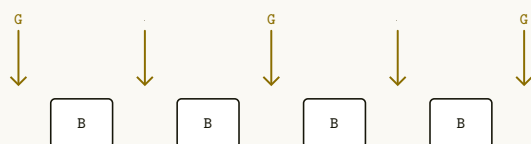
$$\text{no 7 : } 8 \cdot 9 \cdot 9 \cdot 9 = 5832$$

$$\text{at least one 7 : } 9000 - 5832 = 3168$$

ANSWER 3168

1.6 TOGETHER, AND NEVER TOGETHER

- (i) **Together:** tie the k objects into one block. Arrange the $n - k + 1$ units, then the k objects inside the block: $(n - k + 1)! k!$.
- (ii) **No two together:** arrange the other p objects first. They leave $p + 1$ gaps, and each of the m objects to be kept apart takes its own gap.



4 boys make 5 gaps; seat the girls in 3 of them: $4! \binom{5}{3} 3!$

EXAMPLE 1.5

In how many arrangements of the letters of BANANA are the two N's not next to each other?

First arrange B, A, A, A in a row (three alike):

$$\frac{4!}{3!} = 4 \text{ ways, such as } B A A A$$

These leave 5 gaps, including the two ends:

$$\square B \square A \square A \square A \square$$

The two N's go into two different gaps, so they are never next to each other. The N's are alike, so we only choose 2 of the 5 gaps:

$$\binom{5}{2} = 10$$

Total = $4 \cdot 10 = 40$.

ANSWER 40

WHERE MARKS GO · TWO DIFFERENT CONDITIONS

"Not all together" is the total minus the all-together count. "No two together" is the gap count. For three or more objects these are different numbers.

1.7 NUMBERS WITH DIGIT CONDITIONS

- (i) The first digit is never 0. Fill the first place first, or count all strings and subtract those that begin with 0.
- (ii) For divisibility by 2, 5 or 10, fix the last digit first; for 4, the last two digits.
- (iii) Arranging n distinct non-zero digits in all ways, each digit stands in each place $(n - 1)!$ times. With 1, 2, 3 the six numbers add to $2! \cdot (1 + 2 + 3) \cdot 111 = 1332$.

EXAMPLE 1.6

How many 4-digit numbers with all digits different are divisible by 5?

Divisible by 5, so the last digit is 0 or 5.

Last digit 0: the first three are different digits from 1 to 9 (0 is used):

$$9 \cdot 8 \cdot 7 = 504$$

Last digit 5: first digit not 0, not 5, so 8 choices. The two middle digits: 8 choices, then 7.

$$8 \cdot 8 \cdot 7 = 448$$

Total = $504 + 448 = 952$.

ANSWER 952

1.8 OVERCOUNTING

- (i) Count in an ordered way, then divide by k if every object was counted exactly k times. Handshakes among n people: $\frac{n(n-1)}{2}$.
- (ii) If objects are counted different numbers of times, sort them into groups by that number, or fix one rule that builds each object exactly once.

WHERE MARKS GO · CHECK BEFORE DIVIDING

In the next example, dividing 600 by 2 is wrong: some numbers come from one order, some from two and some from four. Find how many times each object is counted before dividing.

EXAMPLE 1.7

N is the number of different 8-digit numbers obtained by arranging the six cards 0, 1, 2, 3, 10, 23, where the first digit is not zero. Find the sum of the digits of N . (IOQM 2026, Q22)

Orders of the six cards: $6! = 720$. Of these, $5! = 120$ start with the card 0, which is not allowed. So

$$720 - 120 = 600 \text{ orders}$$

But two different orders can give the same number:

$$1|0|2|3|10|23 \rightarrow 10231023$$

$$10|2|3|1|0|23 \rightarrow 10231023$$

Card 1 just before card 0 reads 10, the same as card 10; swapping them gives the same number. Similarly, 2 just before 3 can swap with card 23. No other cards together read as one card, so these are the only repeats.

Orders with these pairs (tie each pair into a block):

$$1 \text{ just before } 0 : 5! = 120$$

$$2 \text{ just before } 3 : 5! - 4! = 96$$

$$\text{both} : 4! = 24$$

In the second line, the $4!$ orders that start with 0 are removed.

Grouping the orders that give the same number:

	orders	group	numbers
both pairs	24	4	6
only 1,0	96	2	48
only 2,3	72	2	36
no pair	408	1	408

Here $96 = 120 - 24$, $72 = 96 - 24$ and $408 = 600 - 24 - 96 - 72$.

$$N = 6 + 48 + 36 + 408 = 498$$

$$\text{ANSWER } 4 + 9 + 8 = 21$$

EXAMPLE 1.8

A six-digit number is given. How many seven-digit numbers become the given number when one of their digits is removed? (Leningrad 1982)

Let us take the number 123456 as an example. Such a seven-digit number comes from putting one new digit into one of the 7 gaps:

$$\square 1 \square 2 \square 3 \square 4 \square 5 \square 6 \square$$

This gives at most $7 \cdot 10 = 70$ numbers, but some come twice: a 3 put just before or just after the 3 gives the same number 1233456.

So we use one rule: never put a digit just before an equal digit. Then every number is made exactly once.

Each of the 6 gaps before a digit has 9 choices (not the digit after the gap), and the last gap has 10:

$$6 \cdot 9 + 10 = 64$$

One of these puts 0 in front, which is not allowed:

$$64 - 1 = 63$$

The same count holds for every six-digit number (its first digit is never 0).

ANSWER 63

EXERCISES 1

- 1.1 How many ordered pairs of integers (x, y) satisfy $0 < |xy| < 36$? (Math Prize for Girls 2009)
- 1.2 How many even 4-digit numbers have all their digits different?
- 1.3 In how many arrangements of the letters of ALGEBRA are the two A's not next to each other?
- 1.4 Find the sum of all 4-digit numbers formed by using each of 1, 2, 3, 4 once.
- 1.5 In how many ways can 8 people sit at a round table if two of them must not sit together?
- 1.6 How many necklaces can be made from 6 different beads?
- 1.7 In how many ways can 3 numbers be chosen from $1, 2, \dots, 20$ so that their sum is even?
- 1.8 Find $\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \dots + \binom{10}{3}$.

- 1.9 A committee of 5 is chosen from 6 men and 4 women. How many committees have at least 2 women?
- 1.10 How many rectangles, of all sizes, are formed by the lines of an 8×8 chessboard?
- 1.11 a, b, c are three different numbers from $\{-3, -2, -1, 0, 1, 2, 3\}$, and the line $ax + by + c = 0$ makes an acute angle with the positive x axis. How many different lines are there? (China 1999)

- 1.12 A grid 3 squares high and 4 wide needs 31 unit matchsticks. Grids a high and b wide with $a < b$ are made using exactly 337 matchsticks each. Find the sum of the areas of all such grids. (Crux Mathematicorum)
- 1.13 An $M \times N$ grid of unit squares, $2 \leq M \leq N$, contains exactly 100 non-congruent rectangles with sides along the grid lines. Find all such grids. (Australian Intermediate 2021)

CHAPTER 02 DISTRIBUTIONS AND BIJECTIONS

Many counting questions are about sharing: identical sweets among children, different books among students, a number written as a sum. The first thing to decide is what is identical and what is distinct, among the objects and among the boxes. This chapter covers stars and bars, upper limits on the parts, distinct objects in boxes, lattice paths, Catalan numbers, bijections, compositions and partitions.

2.1 STARS AND BARS

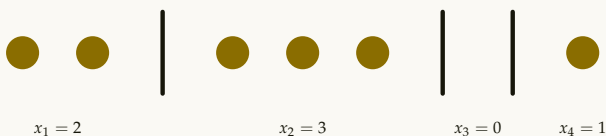
THEOREM · STARS AND BARS

The equation $x_1 + x_2 + \cdots + x_k = n$ has

$$\binom{n+k-1}{k-1} \text{ solutions with all } x_i \geq 0,$$

$$\binom{n-1}{k-1} \text{ solutions with all } x_i \geq 1.$$

A solution in non-negative integers is a row of n stars and $k-1$ bars: x_1 is the number of stars before the first bar, x_2 the number between the first two bars, and so on. Choosing the places of the bars among the $n+k-1$ places fixes the solution.



9 places, choose where the 3 bars go: $\binom{9}{3} = 84$ solutions

- (i) The same number counts n identical objects in k distinct boxes.
- (ii) A lower limit $x_i \geq a_i$ is removed by putting $y_i = x_i - a_i$.
- (iii) $x_1 + \cdots + x_k \leq n$ becomes an equation with one

extra variable $x_{k+1} = n - (x_1 + \cdots + x_k) \geq 0$: $\binom{n+k}{k}$ solutions.

EXAMPLE 2.1

In how many ways can 20 identical chocolates be shared among 4 children so that each child gets at least 3?

Give each child 3 chocolates first: 12 used, 8 left, to be shared freely. Let the children get y_1, y_2, y_3, y_4 more (a child may get none).

$$y_1 + y_2 + y_3 + y_4 = 8, \quad y_i \geq 0$$

By stars and bars (8 stars, 3 bars),

$$\binom{11}{3} = 165$$

ANSWER 165

2.2 UPPER LIMITS ON THE PARTS

For $x_1 + \cdots + x_k = n$ with $0 \leq x_i \leq c$, subtract the solutions in which some $x_i \geq c+1$, add back those in which two are, and so on (Chapter 3). With $m_j = n - j(c+1)$, the count is

$$\sum_{j \geq 0} (-1)^j \binom{k}{j} \binom{m_j + k - 1}{k - 1},$$

stopping when m_j becomes negative. For example, three dice show a total of 10 in $\binom{9}{2} - 3\binom{6}{2} = 36 - 9 = 27$ ways: put $y_i = x_i - 1$, so $y_1 + y_2 + y_3 = 7$ with each $y_i \leq 5$.

When there are only two or three variables, it is often quicker to count directly: two digits add to s in $s+1$ ways for $0 \leq s \leq 9$, and in $19-s$ ways for $10 \leq s \leq 18$.

EXAMPLE 2.2

N polynomials of degree at most 3 have every coefficient a non-negative integer less than 10 and $P(-1) = -9$. Find $N/4$. (Hong Kong Preliminary 2018)

Let $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, each a_i a digit from 0 to 9.

$$P(-1) = -a_3 + a_2 - a_1 + a_0 = -9$$

$$\text{so } (a_1 + a_3) - (a_0 + a_2) = 9$$

Let $a_0 + a_2 = v$. Then $a_1 + a_3 = v + 9 \leq 18$, so $v = 0, 1, \dots, 9$.

Two digits add to s in $s + 1$ ways for $s \leq 9$ (sum 2: $0 + 2, 1 + 1, 2 + 0$) and in $19 - s$ ways for $s \geq 10$. So (a_0, a_2) has $v + 1$ choices and (a_1, a_3) has $19 - (v + 9) = 10 - v$:

v	0	1	2	3	4
ways	$1 \cdot 10$	$2 \cdot 9$	$3 \cdot 8$	$4 \cdot 7$	$5 \cdot 6$

For $v = 5, \dots, 9$ we get the same products in reverse order.

$$\therefore N = 2(10 + 18 + 24 + 28 + 30) = 220$$

ANSWER $N/4 = 55$

2.3 DISTINCT OBJECTS INTO BOXES

- (i) n distinct objects into k distinct boxes: each object picks its box, k^n ways.
- (ii) At most one object in each box needs $n \leq k$, and gives $P(k, n)$ ways.
- (iii) Every box non-empty: by inclusion and exclusion (Chapter 3),

$$k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \dots$$

- (iv) Into k identical non-empty boxes: divide the last count by $k!$, as the $k!$ namings of the boxes give the same split. This is $S(n, k)$; for example $S(n, 2) = 2^{n-1} - 1$.

EXAMPLE 2.3

In how many ways can 5 different books be given to 3 children so that every child gets at least one book?

Let the children be A, B, C . Each book goes to any child: $3^5 = 243$ ways. Remove the ways in which some child gets nothing.

A gets nothing: each book goes to B or C , $2^5 = 32$ ways. Similarly for B and C , so remove $3 \cdot 32 = 96$.

A way with two children getting nothing (all books to one child) was removed twice. There are 3 such ways; add them back once.

$$243 - 96 + 3 = 150$$

ANSWER 150

WHERE MARKS GO · IDENTICAL OR DISTINCT

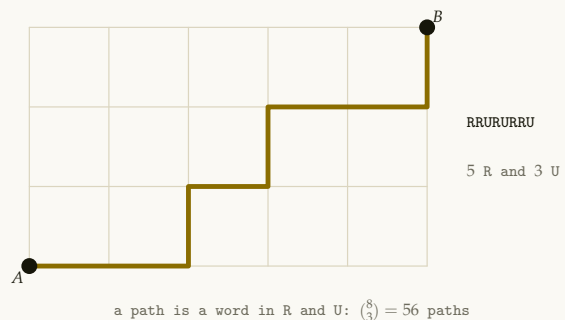
Stars and bars counts identical objects. For distinct objects the count is k^n , or the formula above. Decide this before choosing a formula.

2.4 LATTICE PATHS

THEOREM · PATHS ON A GRID

Paths from $(0, 0)$ to (m, n) made of unit steps right (R) and up (U) are words with m letters R and n letters U, so there are

$$\binom{m+n}{m}.$$



- (i) **Through a point P :** multiply the number of paths to P by the number from P . **Avoiding P :** subtract this from the total.
- (ii) With blocked points or other moves, write at each point the number of ways to reach it: it is the sum of the numbers at the points that lead into it.

EXAMPLE 2.4

A restricted knight moves either 1 up and 2 right, or 2 up and 1 right. How many paths take it from the lower left corner to the upper right corner of a 10×10 board? (ARML 2013)

Let the moves be X (2 right, 1 up) and Y (1 right, 2 up). From corner to corner the knight goes 9 squares right and 9 up. With a moves X and b moves Y ,

$$\text{right: } 2a + b = 9, \quad \text{up: } a + 2b = 9$$

On subtracting, $a = b$; then $3a = 9$, so $a = b = 3$.

So a path is 6 moves, three X and three Y in any order, such as $XYXYXY$ (the knight moves only right and up, so it never leaves the board). Choosing the places of the three X 's,

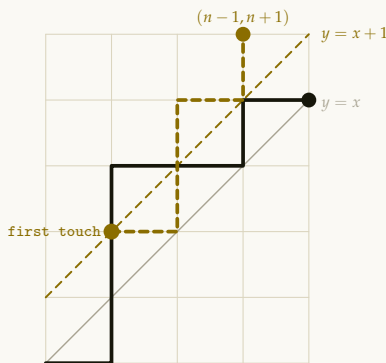
$$\binom{6}{3} = 20$$

ANSWER 20**2.5 CATALAN NUMBERS****THEOREM · CATALAN NUMBERS**

The paths from $(0, 0)$ to (n, n) that never go above the line $y = x$ number

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1} \binom{2n}{n}.$$

A bad path touches the line $y = x + 1$. Reflect the part of the path after the first touch in that line. This gives a path to $(n-1, n+1)$, and every path to $(n-1, n+1)$ arises from exactly one bad path, so the bad paths number $\binom{2n}{n+1}$.



reflect the rest of a bad path in $y = x + 1$: bad paths to (n, n) match all paths to $(n-1, n+1)$

$$(i) C_0, C_1, C_2, \dots = 1, 1, 2, 5, 14, 42, 132, 429, \dots$$

(ii) C_n also counts balanced arrangements of n pairs of brackets, ways to cut a convex $(n+2)$ -gon into triangles by diagonals, and orders of counting n votes for each of two candidates in which the first is never behind.

EXAMPLE 2.5

Shortest paths along the edges of a 7×7 grid go from the bottom left corner to the top right corner. N of them have no edge above the diagonal joining these corners. If $N = 100p + q$ with $0 < q < 100$, find q . (Australian Intermediate 2015)

A shortest path has 7 moves right (R) and 7 moves up (U). For a 2×2 grid, the paths that never go above the diagonal are

$$RRUU \quad \text{and} \quad RURU$$

i.e., $2 = C_2$. Similarly, for the 7×7 grid the number is the Catalan number C_7 :

$$N = \frac{1}{8} \binom{14}{7} = \frac{3432}{8} = 429$$

$429 = 4 \cdot 100 + 29$, so $q = 29$.

ANSWER 29**2.6 BIJECTIONS**

To count a set, match it one to one with a set that is easier to count.

- (i) Subsets of an n -element set match strings of 0s and 1s of length n : 2^n .
- (ii) Choosing r numbers from $1, \dots, n$ with no two consecutive: subtract $0, 1, \dots, r-1$ from them in increasing order. This gives any r numbers from $1, \dots, n-r+1$, so the count is $\binom{n-r+1}{r}$.

EXAMPLE 2.6

A triple (a, b, c) from $\{1, 2, \dots, 17\}$ is pleasing if $b - a$ and $c - b$ are both greater than 3 and at least one of them equals 4. How many pleasing triples are there? (Math Prize for Girls 2010)

Both gaps are more than 3, so both are at least 4. From the triples with both gaps at least 4, remove those with both gaps at least 5; the rest have at least one gap exactly 4.

Gaps at least 4. Put $x = a, y = b - 3, z = c - 6$, e.g. $(1, 5, 9) \rightarrow (1, 2, 3)$. Then $x < y < z$ are three numbers from 1 to 11, and every such choice gives back a triple $(x, y + 3, z + 6)$:

$$\binom{11}{3} = 165$$

Gaps at least 5. Put $x = a, y = b - 4, z = c - 8$: three numbers from 1 to 9,

$$\binom{9}{3} = 84$$

Hence $165 - 84 = 81$ pleasing triples.

ANSWER 81

WHERE MARKS GO · BOTH DIRECTIONS

A bijection needs a rule in both directions. Say how each object is sent to the new set, and how it comes back.

2.7 COMPOSITIONS

A **composition** of n is a way of writing n as an ordered sum of positive parts; $3 = 2 + 1$ and $3 = 1 + 2$ are different. By stars and bars there are $\binom{n-1}{k-1}$ with k parts, so 2^{n-1} in all.

If the parts must come from a set P , split on the first part:

$$g(m) = \sum_{p \in P} g(m - p),$$

with $g(0) = 1$ and $g(m) = 0$ for $m < 0$.

EXAMPLE 2.7

Find the number of positive integers n such that the digits of n lie in $\{1, 2, 4, 8\}$, the digits add to 14, and a digit 1 can occur only immediately to the right of an 8. (IOQM 2026, Q7)

Each 1 has an 8 immediately before it; join the two into a block 81. So n is a row of the blocks

$$2, \quad 4, \quad 8, \quad 81$$

with digit sums 2, 4, 8, 9 adding to 14 (811 is not allowed: its second 1 has a 1 before it).

No block 81. One block 81 leaves $14 - 9 = 5$ for the other blocks, which is odd; but 2, 4, 8 are even. Two or more blocks 81 give at least $18 > 14$.

So n uses only 2, 4, 8, with sum 14. Halving every digit, we need ordered lists of 1, 2, 4 with sum 7. Let $g(m)$ be the number of such lists with sum m . The first entry is 1, 2 or 4, so

$$g(m) = g(m-1) + g(m-2) + g(m-4), \quad g(0) = 1$$

with $g(m) = 0$ for $m < 0$.

m	0	1	2	3	4	5	6	7
$g(m)$	1	1	2	3	6	10	18	31

Check: $g(7) = g(6) + g(5) + g(3) = 18 + 10 + 3$.

ANSWER 31

2.8 PARTITIONS

A **partition** of n is a way of writing n as an unordered sum of positive parts. List each one with its parts in decreasing order, so that it is written once.

(i) $p(1), \dots, p(10) = 1, 2, 3, 5, 7, 11, 15, 22, 30, 42$.

(ii) Partitions into at most k parts are as many as partitions into parts of size at most k .

(iii) **Euler:** partitions into distinct parts are as many as partitions into odd parts.

EXAMPLE 2.8

In how many ways can Rs. 10 be paid using coins of Rs. 1, 2 and 5? (Only the number of coins of each kind matters.)

Count by the number of Rs. 5 coins: 0, 1 or 2 (as $3 \cdot 5 > 10$).

Two coins of Rs. 5: nothing is left, 1 way.

One coin of Rs. 5: Rs. 5 is left: 0, 1 or 2 coins of Rs. 2, the rest in Rs. 1 coins, 3 ways.

No coin of Rs. 5: 0, 1, $\dots$, 5 coins of Rs. 2, the rest in Rs. 1 coins, 6 ways.

$$1 + 3 + 6 = 10$$

ANSWER 10

EXERCISES 2

2.1 How many solutions in non-negative integers does $x_1 + x_2 + x_3 + x_4 = 15$ have?

- 2.2 How many solutions in positive integers does $x + y + z + w = 12$ have?
- 2.3 How many solutions in non-negative integers does $x + y + z \leq 10$ have?
- 2.4 In how many ways can 12 identical balls be put into 4 different boxes with at most 5 balls in each box?
- 2.5 How many 4-digit numbers have digit sum 10?
- 2.6 In how many ways can 6 different books be given to 3 children so that each child gets at least one?
- 2.7 In how many ways can 6 people be split into 3 non-empty groups, the groups not being named?
- 2.8 How many shortest grid paths go from $(0,0)$ to $(5,5)$ without passing through $(2,2)$?
- 2.9 In how many ways can 6 votes for A and 6 votes for B be counted one by one so that A is never behind B ?
- 2.10 In how many ways can 4 numbers be chosen from $1, 2, \dots, 15$ with no two of them consecutive?
- 2.11 In how many ways can 10 be written as an ordered sum of positive integers, each part at least 2?
- 2.12 How many partitions of 10 have at most 3 parts?
- 2.13 How many compositions does 10 have?

CHAPTER 03 INCLUSION, EXCLUSION AND RECURSION

Some conditions are hard to count directly. “None of these happens” is counted by inclusion and exclusion, and a condition on neighbouring places is counted by a recursion, building the object one place at a time. This chapter covers both, together with derangements, strings that must use given values, colourings of rows and rings, and counting stage by stage.

3.1 INCLUSION AND EXCLUSION

THEOREM · INCLUSION AND EXCLUSION

$$\begin{aligned}
 |A \cup B| &= |A| + |B| - |A \cap B|, \\
 |A \cup B \cup C| &= |A| + |B| + |C| \\
 &\quad - |A \cap B| - |B \cap C| - |C \cap A| \\
 &\quad + |A \cap B \cap C|.
 \end{aligned}$$

For more sets: add the single sets, subtract the pairs, add the triples, and so on.

An element that lies in exactly $t \geq 1$ of the sets is counted $\binom{t}{1} - \binom{t}{2} + \binom{t}{3} - \dots = 1$ time, which is why the formula is right. The number of elements in *none* of the sets is the total minus the union.

For example, the integers from 1 to 1000 divisible by none of 2, 3, 5 number

$$\begin{aligned}
 1000 - (500 + 333 + 200) \\
 + (166 + 100 + 66) - 33 = 266.
 \end{aligned}$$

EXAMPLE 3.1

Three students together solved exactly 100 problems, and each of them solved exactly 60. A problem is difficult if only one of them solved it, and easy if all three did. Find $d - e$, where d and e are the numbers of difficult and easy problems. (Leningrad 1990)

Let x_1, x_2, x_3 be the numbers of problems solved by exactly 1, 2, 3 students. Then $d = x_1$ and $e = x_3$. Counting problems,

$$x_1 + x_2 + x_3 = 100$$

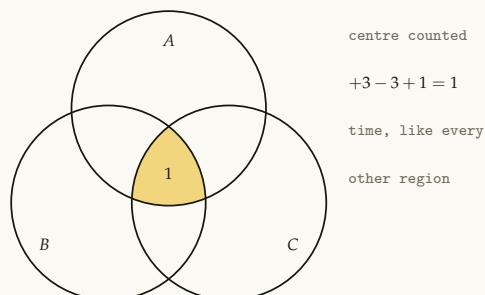
Counting solutions, the students solved $3 \cdot 60 = 180$ in all (a problem solved by 2 students is counted twice, by 3 students three times):

$$x_1 + 2x_2 + 3x_3 = 180$$

Subtracting the second equation from twice the first,

$$x_1 - x_3 = 200 - 180 = 20$$

ANSWER 20



3.2 DERANGEMENTS

A **derangement** is an arrangement in which no object is in its own place.

THEOREM · DERANGEMENTS

$$D_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots + \frac{(-1)^n}{n!} \right).$$

- (i) $D_1, \dots, D_7 = 0, 1, 2, 9, 44, 265, 1854$.
 (ii) $D_n = (n-1)(D_{n-1} + D_{n-2})$.
 (iii) Exactly k objects in their own places: $\binom{n}{k} D_{n-k}$.

EXAMPLE 3.2

Seven students exchange their test papers for marking. In how many ways can the papers be given out so that exactly two students mark their own?

The 2 students who mark their own papers are chosen in

$$\binom{7}{2} = 21 \text{ ways}$$

The other 5 papers all go to wrong students, a derangement of 5 objects: $D_5 = 44$ ways. Hence

$$21 \cdot 44 = 924$$

ANSWER 924

3.3 STRINGS THAT USE GIVEN VALUES

- (i) Strings of length n from k values that use every value: $k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \dots$.
 (ii) Strings from $w \geq 2$ values that use two particular values:

$$w^n - 2(w-1)^n + (w-2)^n.$$

- (iii) For any three real numbers, $|x-y| + |y-z| + |z-x| = 2(\max - \min)$: the middle number cancels.

EXAMPLE 3.3

Find the number of ordered triples (x, y, z) of integers with $1 \leq x, y, z \leq 8$ and $|x-y| + |y-z| + |z-x| = 8$. (IOQM 2026, Q17)

Putting $x = 2, y = 5, z = 6$,

$$|2-5| + |5-6| + |6-2| = 3 + 1 + 4 = 8 = 2(6-2)$$

In general, let the three numbers in order be $m \leq t \leq M$. The sum is

$$(t-m) + (M-t) + (M-m) = 2(M-m)$$

So $M-m = 4$, and $m = 1, 2, 3, 4$ (as $M = m+4 \leq 8$).

For $m = 1$, all three numbers lie in $1, \dots, 5$, and both 1 and 5 appear. From all 5^3 triples, remove those without 1 and those without 5, and add back those without both:

$$125 - 64 - 64 + 27 = 24$$

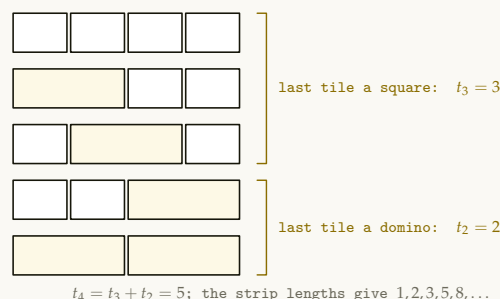
Similarly, each of the 4 values of m gives 24 (add $m-1$ to every number).

ANSWER $4 \cdot 24 = 96$

3.4 SETTING UP A RECURSION

KEY IDEA · SPLIT ON THE LAST PIECE

Let a_n be the number of objects of size n . Split them by their last piece (or first piece). Each case is an object of smaller size followed by that piece, so a_n is a sum of earlier terms.



Tilings of a $1 \times n$ strip by squares and dominoes end in a square or a domino, so

$$t_n = t_{n-1} + t_{n-2}, \quad t_1 = 1, t_2 = 2.$$

Strings of 0s and 1s of length n with no two adjacent 1s follow the same rule, with $f_1 = 2, f_2 = 3$.

WHERE MARKS GO · TEST SMALL CASES

Before using a recursion, list the cases $n = 1, 2, 3$ by hand and check that the rule gives them. A missing or doubled case shows up at once.

EXAMPLE 3.4

A raga has short notes of 1 beat and long notes of 2 beats. There are 3 ragas of 3 beats. How many ragas are 11 beats long? (Maryland HSMC 2023)

Let r_n be the number of ragas of n beats, S for short and L for long. For $n = 3$ they are SSS, SL, LS.

If the last note is short, the rest is a raga of $n - 1$ beats; if it is long, the rest is a raga of $n - 2$ beats. So

$$r_n = r_{n-1} + r_{n-2}, \quad r_1 = 1, r_2 = 2$$

Adding the last two values each time,

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144$$

ANSWER $r_{11} = 144$

3.5 RECURSIONS WITH STATES

When the allowed next piece depends on the last one, keep one count for each possible last piece. These counts are the *states*, and each is found from the states one place earlier.

EXAMPLE 3.5

How many 5-digit numbers use only the digits 1, 2, 3, with neighbouring digits differing by at most 1? (Crux Mathematicorum)

After a 1 the next digit is 1 or 2; after a 2, any digit; after a 3, 2 or 3. By last digit, the numbers with n digits are

n	1	2	3	4	5
end 1	1	2	5	12	29
end 2	1	3	7	17	41
end 3	1	2	5	12	29

Each column comes from the one before it:

$$\text{end 1} = (\text{end 1}) + (\text{end 2})$$

$$\text{end 2} = \text{all three}$$

$$\text{end 3} = (\text{end 2}) + (\text{end 3})$$

For $n = 5$: $29 + 41 + 29 = 99$.

ANSWER 99

3.6 CLASSIC RECURSIONS

(i) **Tower of Hanoi:** $h_n = 2h_{n-1} + 1$, so $h_n = 2^n - 1$.

(ii) **Lines:** n lines, no two parallel and no three through a point, cut the plane into $r_n = r_{n-1} + n$ regions, so $r_n = \frac{n^2+n+2}{2}$. The n th line meets the others at $n - 1$ points and is cut into n pieces, each splitting a region.

(iii) **Dominoes:** a $2 \times n$ board is tiled by 1×2 dominoes in $d_n = d_{n-1} + d_{n-2}$ ways, with $d_1 = 1$, $d_2 = 2$.

(iv) **Derangements:** $D_n = (n - 1)(D_{n-1} + D_{n-2})$.

A recursion $a_n = pa_{n-1} + qa_{n-2}$ is solved through the roots α, β of $x^2 = px + q$: $a_n = A\alpha^n + B\beta^n$ when $\alpha \neq \beta$.

3.7 COLOURING ROWS AND RINGS

(i) n regions in a row, k colours, neighbours different: $k(k - 1)^{n-1}$.

(ii) $n \geq 3$ regions in a **ring**, neighbours different:

$$(k - 1)^n + (-1)^n(k - 1).$$

For the ring, colour it as if it were a row, $k(k - 1)^{n-1}$ ways, and remove the colourings whose first and last regions agree; these are ring colourings with $n - 1$ regions. So $R_n = k(k - 1)^{n-1} - R_{n-1}$, and $R_3 = k(k - 1)(k - 2)$ gives the formula.

EXAMPLE 3.6

The 6 sectors of a ring are coloured with 3 colours so that neighbouring sectors have different colours. In how many ways can this be done?

Putting $n = 6$, $k = 3$ in the ring formula,

$$(3 - 1)^6 + (-1)^6(3 - 1) = 64 + 2 = 66$$

Check: $R_n = 3 \cdot 2^{n-1} - R_{n-1}$ with $R_3 = 3 \cdot 2 \cdot 1 = 6$ gives

$$R_4 = 24 - 6 = 18, \quad R_5 = 48 - 18 = 30$$

$$R_6 = 96 - 30 = 66$$

ANSWER 66

3.8 COUNTING STAGE BY STAGE

When every stage offers the same number of choices, whatever happened before, multiply, even if the choices themselves change from case to case.

EXAMPLE 3.7

A 1×5 rectangle is divided into five unit squares. Each of the 16 unit segments is coloured red, blue or green. A square is colourful if its four sides use all three colours. If N is the number of colourings in which all five squares are colourful, find the remainder when N is divided by 100. (IOQM 2026, Q23)

Let the 6 vertical segments be $v_0, \dots, v_5$ from left to right. Square i has left side v_{i-1} , right side v_i , top t_i and bottom b_i .

Colour from left to right: v_0 in 3 ways.

In square 1, v_0 is already coloured, say red, so the square is colourful when v_1, t_1, b_1 together use blue and green. Of the $3^3 = 27$ colourings of these three, $2^3 = 8$ have no blue and 8 have no green; the 1 with neither is removed twice:

$$27 - 8 - 8 + 1 = 12$$

Similarly, every square gives 12, whatever the colour of its left side. So

$$N = 3 \cdot 12^5 = 746496$$

ANSWER 96

EXAMPLE 3.8

The vertices of a regular octagon are labelled $1, 2, \dots, 8$. A labelling is good if the path $1 \rightarrow 2 \rightarrow \dots \rightarrow 8$ along straight segments does not cross itself. If N is the number of good labellings, find N modulo 100. (IOQM 2026, Q18)

Label 1 can go on any of the 8 vertices.

After labels $1, \dots, k$, the unused vertices form one arc of the circle, and label k is next to an end of this arc.

If label $k+1$ is inside the arc, the segment from k to $k+1$ divides the octagon into two parts, both with unused vertices. The path must visit both parts later, so it crosses this segment, which is not good.

So label $k+1$ goes to one of the two ends of the arc, and the unused vertices again form an arc. Each such choice is good: no two used vertices and no two unused vertices lie on opposite sides of the new segment, so it crosses no earlier or later segment. Labels $2, \dots, 7$ have 2 choices each, and label 8 takes the last vertex:

$$N = 8 \cdot 2^6 = 512$$

ANSWER 12

WHERE MARKS GO · WHEN THE CHOICES CHANGE

If the number of choices at a stage depends on what came before, multiplying is wrong. Split into cases, or keep states as in 3.5.

EXERCISES 3

- 3.1 A and B are sets, and exactly 144 sets are subsets of A or of B . How many elements does $A \cup B$ have? (*Crux Mathematicorum*)
- 3.2 How many integers from 1 to 1000 are divisible by 2 or 3 but not by 5?
- 3.3 In how many ways can 6 letters be put in their 6 addressed envelopes so that no letter is in its own envelope?
- 3.4 In how many arrangements of $1, 2, \dots, 7$ are exactly 3 numbers in their own places?
- 3.5 How many strings of length 5 from the letters a, b, c use all three letters?
- 3.6 How many strings of 6 decimal digits contain each of $1, 2, 3$ at least once?
- 3.7 In how many ways can a 1×10 strip be tiled with 1×1 squares and 1×2 dominoes?
- 3.8 How many strings of 0s and 1s of length 10 have no two adjacent 1s?
- 3.9 A regular 9-gon is $ABCDEFGHI$. A trip is a sequence of 4 of its letters with no two neighbouring letters equal. How many trips start at A and end at B ? (*ARML 2004*)
- 3.10 Two teams of 4 players play in a fixed order: the loser of each game leaves and the winner meets the next player of the other team, until one team has no players left. How many different sequences of games are possible? (*China 1988*)

- 3.11 In how many ways can the 6 sectors of a ring be coloured with 4 colours so that neighbouring sectors differ?
- 3.12 Into how many regions do 10 lines cut the plane, if no two are parallel and no three pass through one point?

- 3.13 The numbers $1, \dots, 6$ are arranged in a row, and each row above is formed by adding neighbouring pairs, until one number is left at the top. For how many arrangements is the top number a multiple of 5? (*Australian Intermediate 2022*)

CHAPTER 04 PIGEONHOLE, COLOURING AND INVARIANTS

Not every combinatorics question is a count. Many ask for a largest or least number: the least number of objects that forces something to happen, or the most objects that can be placed under a rule. Such an answer has two parts, a bound and an example that reaches it. This chapter covers the pigeonhole principle, colouring arguments, invariants, monovariants, and the extremal and greedy principles.

4.1 THE PIGEONHOLE PRINCIPLE

THEOREM · PIGEONHOLE

If N objects are put into k boxes, some box holds at least $\lceil \frac{N}{k} \rceil$ objects. In particular, $k + 1$ objects in k boxes put two objects in one box.

The counting is easy; the work is in choosing the boxes. For example, any 5 points in a square of side 1 include two at distance at most $\frac{\sqrt{2}}{2}$: cut the square into four squares of side $\frac{1}{2}$, and two of the points lie in the same small square.

A count of this kind also gives upper bounds. If every box can hold at most c objects, there are at most kc objects.

EXAMPLE 4.1

Each of six competitions has seven problems, and in each competition exactly four problems appear in no other competition. What is the greatest number of different problems in the six competitions? (Leningrad 1989)

Each competition has 4 problems of its own: $6 \cdot 4 = 24$ problems.

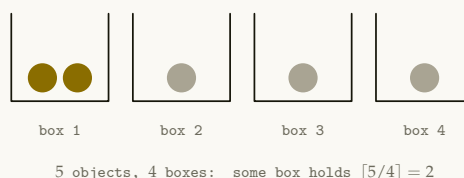
Each competition has 3 more problems, each also in another competition: $6 \cdot 3 = 18$ places. A shared problem fills at least 2 of these places, so there are at most $18/2 = 9$ shared problems. Hence, in all, at

most

$$24 + 9 = 33$$

Example with 33. Let the competitions be P_1, P_2, P_3 and Q_1, Q_2, Q_3 . For each pair P_i, Q_j take one common problem: 9 shared problems, 3 in each competition. Add 4 own problems to each: $3 + 4 = 7$ in each.

ANSWER 33



4.2 CHOOSING THE BOXES

- (i) Any $n + 1$ numbers from $1, \dots, 2n$ include two consecutive numbers: use the boxes $\{1, 2\}, \{3, 4\}, \dots$
- (ii) Any $n + 1$ numbers from $1, \dots, 2n$ include one that divides another: put each number in the box of its largest odd divisor. Two numbers in one box differ by a power of 2.
- (iii) Any n integers have a block of consecutive terms whose sum is a multiple of n : among the $n + 1$ sums $S_0 = 0, S_1, \dots, S_n$ of the first terms, two leave the same remainder.
- (iv) Any 5 points with integer coordinates include two whose midpoint has integer coordinates: use the 4 boxes given by the parities of x and y .

WHERE MARKS GO · BOUND AND EXAMPLE

For a largest or least value, the bound is half the answer. Show also an example that reaches it, as in the examples of this chapter.

EXAMPLE 4.2

Find the least n such that among any n integers there are two whose sum or difference is a multiple of 10.

Only the last digit matters. Same last digit: the difference ends in 0. Last digits adding to 10, like 3 and 7: the sum ends in 0. So make 6 boxes by the last digit:

$$\{0\}, \{5\}, \{1,9\}, \{2,8\}, \{3,7\}, \{4,6\}$$

Any two numbers in one box work. Hence any 7 numbers have two in one box (pigeonhole).

6 numbers are not enough: take 0, 1, 2, 3, 4, 5. Any two of them have sum from 1 to 9 and difference from 1 to 5, never a multiple of 10.

ANSWER 7

EXAMPLE 4.3

The numbers $1, 2, \dots, 100$ are on a board. What is the least number of them that must be deleted so that no two remaining numbers add up to a prime? (JBMO Shortlist 2018)

At least 50. Pair the numbers as

$$(1, 100), (2, 99), \dots, (50, 51)$$

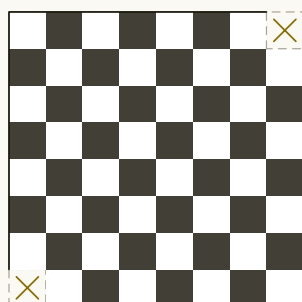
Each pair has sum 101, a prime. So at least one number of each of the 50 pairs must be deleted.

50 are enough. Delete the 50 odd numbers. The sum of two different even numbers is even and greater than 2, so not prime.

ANSWER 50

4.3 COLOURING ARGUMENTS

Colour the board so that every piece covers a fixed pattern of colours. If the numbers of squares of each colour do not match what the pieces can cover, the tiling is impossible.



62 squares left

30 dark, 32 light

a domino covers

one of each

so no tiling

For example, remove two opposite corners from an 8×8 board. Both are dark, so 30 dark and 32 light squares remain. Each domino covers one of each, so the 62 squares cannot be covered by 31 dominoes.

(i) Colour (i, j) by the parity of $i + j$ for dominoes.

(ii) Colour (i, j) by $(i + j) \bmod 3$ for 1×3 pieces.

(iii) For questions about rows and columns, count by rows and by columns.

EXAMPLE 4.4

The central square of a 9×9 chessboard is white. How many white squares are there? (IOQM 2026, Q1)

Let (i, j) be the square in row i , column j . The colours alternate, so the colour of (i, j) depends only on the parity of $i + j$.

Centre $(5, 5)$: $5 + 5 = 10$ is even, and it is white. So the white squares have $i + j$ even, i.e., i, j both odd or both even.

From 1 to 9 there are 5 odd and 4 even numbers. Hence the number of white squares is

$$5 \cdot 5 + 4 \cdot 4 = 25 + 16 = 41$$

ANSWER 41

EXAMPLE 4.5

Checkers are placed on a 7×7 board, at most one in each square, so that every row and every column has an even number of checkers. Find the largest possible number of checkers. (IOQM 2026, Q12)

At most 42. A row has 7 squares and an even number of checkers, so at most 6. Hence 7 rows hold at most $7 \cdot 6 = 42$.

Can we reach 42? Then every row has exactly one empty square: 7 empty squares in all. A column has an even number of checkers, so an odd number of empty squares ($7 - \text{even} = \text{odd}$), so at least 1. 7 columns, 7 empty squares, at least 1 in each: so exactly one in each column.

So leave the 7 squares of a diagonal empty. Then every row and every column has 6 checkers.

ANSWER 42

EXAMPLE 4.6

Each square of an 8×8 board is coloured black or white so that every 3×3 block of squares has an odd number of black squares. Find the least number of black squares. (Saudi Arabia JBMO TST 2019)

At least 4. The four corner 3×3 blocks do not overlap. Each has an odd number of black squares, so at least 1.

4 are enough. Writing (row, column), colour only (3,3), (3,6), (6,3), (6,6) black. Any 3 consecutive rows, such as rows 4,5,6, contain exactly one of rows 3 and 6. Similarly for columns. So every 3×3 block has exactly one black square.

ANSWER 4

4.4 INVARIANTS

DEFINITION · INVARIANT

An **invariant** is a quantity that no allowed move changes. If the starting position and the target differ in it, the target cannot be reached.

Common invariants are the parity of a sum, a sum or product modulo m , and the number of squares of each colour.

EXAMPLE 4.7

The numbers $1, 2, \dots, 10$ are written on a board. A move replaces two numbers a, b by $|a - b|$. After nine moves one number is left. Can it be 0?

For 1,2,3: replacing 1,2 by 1 gives 1,3; then 1,3 by 2. The sum goes $6 \rightarrow 4 \rightarrow 2$: it changes, but stays even. Each move reduces the sum by $(a + b) - |a - b|$, twice the smaller of a, b , so even. Hence the parity of the sum never changes. Initially the sum is $1 + 2 + \dots + 10 = 55$, odd, so the last number is odd and cannot be 0.

ANSWER no, it is always odd

WHERE MARKS GO · WHAT AN INVARIANT PROVES

An invariant shows that a target *cannot* be reached. To show that a target can be reached, give the moves.

4.5 MONOVARIANANTS

A **monovariant** changes in one direction only. If it is a non-negative integer that goes down at every move, the process must stop.

For example, split the vertices of a graph into two groups. While some vertex has more neighbours in its own group than in the other, move it across. Each move increases the number of edges between the groups, which cannot exceed the total number of edges, so the moves stop. Then every vertex has at least half of its neighbours in the other group.

4.6 EXTREMAL AND GREEDY CHOICES

- (i) Look at the largest, smallest, first or last object. It often has a property the others need not have.
- (ii) **Greedy:** to make a total as large as possible, give the largest values to the places that count most.

EXAMPLE 4.8

All six digits of three 2-digit numbers are different. If N is the largest possible sum of three such numbers, find the sum of the digits of N . (IOQM 2026, Q4)

Let the numbers be $\overline{ab}$, $\overline{cd}$, $\overline{ef}$. Their sum is

$$10(a + c + e) + (b + d + f)$$

A tens digit counts 10 times a units digit. So the three largest digits go to the tens places and the next three to the units places (otherwise a swap, or a larger unused digit, gives a larger sum):

$$N = 10(9 + 8 + 7) + (6 + 5 + 4) = 240 + 15 = 255$$

ANSWER $2 + 5 + 5 = 12$

EXERCISES 4

- 4.1 A drawer has 10 red, 10 blue and 10 green socks. How many must be taken out, without looking, to be sure of 3 socks of the same colour?
- 4.2 How many numbers must be chosen from $1, 2, \dots, 100$ to be sure that one of them divides another?

- 4.3 Find the least n such that any n points with integer coordinates include two whose midpoint has integer coordinates.
- 4.4 Find the least n such that any n different numbers in a row contain 4 of them, in order, that increase or that decrease.
- 4.5 Eighteen points are equally spaced on a circle. How many must be chosen to be sure that four of them are the corners of a rectangle? (*Crux Mathematicorum*)
- 4.6 $M = \{1, 2, \dots, 2024\}$, and $A \subseteq M$ has the property that $x \in A$ implies $24x \notin A$. Find the least possible number of elements of M not in A . (*China 1995*)
- 4.7 What is the least number of numbers that must be deleted from $1, 2, \dots, 2024$ so that no remaining number is the product of two other remaining numbers? (*IITian Forum*)
- 4.8 Amy's 14 cousins are aged $2, 3, \dots, 15$. She has cards numbered $16, 17, \dots, k$ and gives each cousin one card whose number is a multiple of that cousin's age. Find the least k for which this is possible. (*AIMO 2022*)
- 4.9 Two opposite corners of an 8×8 board are removed. What is the largest number of dominoes that can be placed on the rest without overlapping?
- 4.10 Knights are placed on a 4×6 board, at most one per square, so that every empty square is attacked. Find the least number of knights. (*ARML 2013*)
- 4.11 The numbers $1, 2, \dots, 20$ are on a board. A move replaces two numbers a, b by $a + b - 1$. Find the number left after 19 moves.
- 4.12 What is the largest number of kings on an 8×8 board, no two attacking each other?
- 4.13 What is the least number of the numbers $1, 2, \dots, 100$ that must be removed so that no remaining number is twice another?

CHAPTER 05 GRAPHS AND PROBLEM SOLVING

This chapter begins with double counting and the small part of graph theory that IOQM uses, then counts points and pieces in polygons and grids. It ends with a chart and a table for choosing the method, the nine combinatorics questions of IOQM 2026, a list of common mistakes and a mixed problem set.

5.1 DOUBLE COUNTING

KEY IDEA · COUNT ONE SET IN TWO WAYS

Count the same collection in two different ways and set the two answers equal.

A **graph** is a set of points (vertices) with some pairs joined (edges). The **degree** of a vertex is the number of edges at it.

THEOREM · HANDSHAKES

Every edge has two ends, so

$$\sum_v \deg(v) = 2E.$$

In particular, the number of vertices of odd degree is even.

So 12 people who each shake hands with exactly 5 others make $\frac{12 \cdot 5}{2} = 30$ handshakes, while 9 people cannot each shake hands with exactly 3 others, since $9 \cdot 3$ is odd. In the same way $\sum_k k \binom{n}{k} = n2^{n-1}$: choose a committee and its leader, or the leader first and then any group of the others.

EXAMPLE 5.1

There are n points in the plane, no three on a line. Every segment joining two of them is red or blue, so that (a) if AB is red and BC is blue, then AC is red, and (b) every point has exactly three red segments. Find the sum of all possible values of n . (IOQM 2026, Q26)

At least 4 points. Each point has 3 red segments, so $n \geq 4$.

At most 6 points. Let A be a point and B one of its 3 red neighbours. Let X be any of the $n - 4$ points other than A and its red neighbours; then AX is blue. Using (a) for B, A, X (BA red, AX blue), BX is red. So B has red segments to A and to all $n - 4$ points X :

$$1 + (n - 4) \leq 3 \Rightarrow n \leq 6$$

Check $n = 4, 5, 6$.

$n = 4$: colour every segment red. Each point has 3 red segments, and (a) never applies (no blue segment). Hence $n = 4$ is possible.

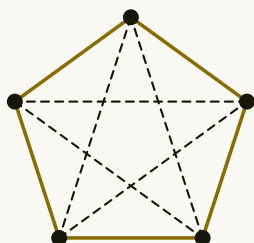
$n = 5$: each point has 4 segments, 3 red and 1 blue. So the blue segments pair up the 5 points, which is not possible (5 is odd).

$n = 6$: split the points into two groups of three; colour segments between the groups red and inside a group blue. Each point has 3 red segments. If AB is red and BC blue, then A, B are in different groups and C is in B 's group, so AC is red. Hence $n = 6$ is possible.

ANSWER $4 + 6 = 10$

5.2 GRAPHS: WHAT TO KNOW

- (i) The complete graph K_n , with every pair joined, has $\binom{n}{2}$ edges. A connected graph with no cycle, a **tree**, on n vertices has $n - 1$ edges.
- (ii) A graph can have its vertices split into two groups with every edge going across exactly when it has no cycle of odd length.
- (iii) **Mantel**: a graph on n vertices with no triangle has at most $\lfloor \frac{n^2}{4} \rfloor$ edges. Two groups as equal as possible, with all edges across, reach it.



pentagon: colour 1
 pentagram: colour 2
 no one colour
 triangle, so $R(3,3) > 5$

EXAMPLE 5.2

Find the least n such that, however the edges of K_n are coloured red and blue, there is a triangle with all three edges of one colour.

6 points are enough. Let v be any point. Its 5 segments have 2 colours, so at least 3 have the same colour, say red, joining v to a, b, c .

If a side of triangle abc , say ab , is red, then vab is a red triangle. If no side is red, then abc is a blue triangle.

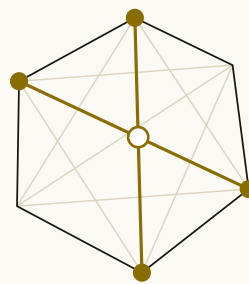
5 points are not enough. Colour the sides of a pentagon red and its diagonals blue (see the figure). The red segments form a pentagon and so do the blue

ones, and a pentagon has no triangle. Deleting points from this colouring shows that no $n < 5$ works either.

ANSWER 6

5.3 COUNTING IN POLYGONS

- (i) A convex n -gon has $\frac{n(n-3)}{2}$ diagonals.
- (ii) If no three diagonals meet at a point, every crossing inside comes from one choice of 4 vertices, so there are $\binom{n}{4}$ crossings.
- (iii) Under the same condition the diagonals cut the polygon into $\binom{n}{4} + \binom{n-1}{2}$ regions.
- (iv) Triangles with vertices among the n vertices: $\binom{n}{3}$. Those with no side along the polygon: $\frac{n(n-4)(n-5)}{6}$.



4 vertices
 $\leftrightarrow$ 1 crossing
 so $\binom{n}{4}$ crossings
 if no three meet

EXAMPLE 5.3

The diagonals of a convex decagon are drawn, and no three of them pass through one point. How many crossing points lie inside the decagon?

Each crossing is on exactly two diagonals (no three meet at a point), with 4 different ends. In the figure, the two gold diagonals of the hexagon cross, and their ends are the 4 gold vertices.

Conversely, any 4 vertices P, Q, R, S , in order round the polygon, give exactly one crossing, of PR and QS . So the number of crossings is the number of choices of 4 vertices:

$$\binom{10}{4} = 210$$

ANSWER 210

5.4 GRIDS AND GEOMETRIC COUNTING

Draw small cases, decide what to count, and then look for a bound. Points of a grid can be split by the parity

of their coordinates, as in Chapter 4.

EXAMPLE 5.4

A square is divided into 25 unit squares. Some diagonals of the unit squares are drawn so that no two of them have a common point. Find the largest possible number of diagonals. (Cuba 2004)

The grid has $6 \cdot 6 = 36$ points. A unit square has at most one diagonal, as its two diagonals cross. No two diagonals have a common point, so k diagonals use $2k$ different points.

At most 17. The top row of points is used only by diagonals in the top row of squares: at most 5, one end each. So at least one of its 6 points is unused. Similarly for the bottom row. Hence $2k \leq 36 - 2$, i.e., $k \leq 17$.

Not 17. If $k = 17$, all points except one in the top row and one in the bottom row are used. Let $d_1, \dots, d_5$ be the numbers of diagonals in the rows of squares, from the top. The top row of points has 5 used points, so $d_1 = 5$. The second row of points is used by rows 1 and 2 of squares, so $d_1 + d_2 = 6$, i.e., $d_2 = 1$. Similarly $d_3 = 5, d_4 = 1, d_5 = 5$. Also, the left and right columns of points each have an unused point, so in the same way square columns 2 and 4 have only 1 diagonal each. But rows 1, 3, 5 are full, so column 2 has at least 3, which is not possible. Hence $k \leq 16$.

16 is possible. Rows of squares from the top; / and \ give the direction, · an empty square:

```

/   ·   \   \   \
/   ·   \   ·   ·
/   /   ·   /   /
·   ·   \   ·   /
\   \   \   ·   /

```

ANSWER 16**5.5 IOQM 2026 COMBINATORICS QUESTIONS**

Nine of the thirty questions of IOQM 2026 were combinatorics, second only to number theory. Each is solved in this part.

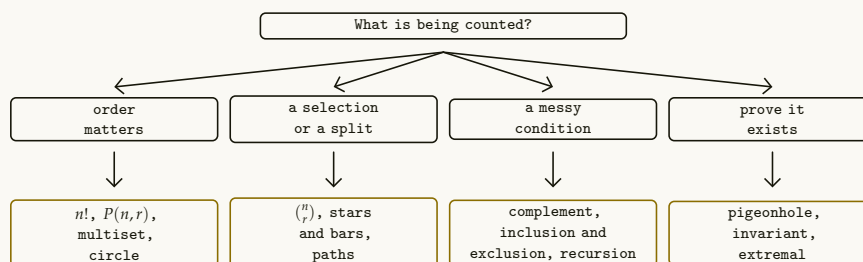
- (i) **Q1**, white squares of a board: parity colouring. Example 4.4.
- (ii) **Q4**, a largest sum: greedy place value. Example 4.8.
- (iii) **Q7**, digit strings: compositions. Example 2.7.
- (iv) **Q12**, checkers with even lines: bound and example. Example 4.5.

- (v) **Q17**, triples with $|x - y| + \dots = 8$: end values.
Example 3.3.
- (vi) **Q18**, labellings of an octagon: stage by stage.
Example 3.8.
- (vii) **Q22**, numbers from cards: unequal overcounting.
Example 1.7.
- (viii) **Q23**, colourful squares: stage by stage.
Example 3.7.

- (ix) **Q26**, a two-coloured graph: double counting.
Example 5.1.

5.6 CHOOSING THE METHOD

A question in the paper does not say which method it wants. Read it once for what is being counted, and use the chart and table below.



THE QUESTION ASKS	METHOD TO TRY FIRST	SECTION
arrangements, order matters	$n!$, $P(n, r)$, alike objects, circles	1.2, 1.4
a selection, order does not matter	$\binom{n}{r}$, complement, gaps	1.3, 1.5, 1.6
identical objects into boxes	stars and bars, upper limits	2.1, 2.2
paths on a grid	$\binom{m+n}{m}$, Catalan, fill in the grid	2.4, 2.5
"none of these"	inclusion and exclusion, derangements	3.1, 3.2
a condition on neighbours	recursion, states, rings	3.4, 3.5, 3.7
"at least" or "always"	pigeonhole, bound and example	4.1, 4.2
a move or a tiling	colouring, invariant	4.3, 4.4

5.7 COMMON MISTAKES

Students often lose marks on these points.

- (i) Adding cases that overlap counts the overlap twice.
- (ii) Multiplying stages whose number of choices depends on earlier choices.
- (iii) Dividing by a symmetry factor when objects are not all counted equally often.
- (iv) Using $(n-1)!$ for a circle of objects that are not all different.
- (v) Mixing up "not all together" and "no two together".
- (vi) Forgetting that a number cannot begin with 0.
- (vii) Stars and bars for distinct objects; it counts identical objects only.

- (viii) Treating an invariant as a proof that a target can be reached.

- (ix) Giving a bound for a maximum without an example that reaches it.

5.8 A MIXED PROBLEM SET

The exercises below are not arranged by chapter, as in the actual paper. Before solving, decide from the table in 5.6 which method the question needs. Answers with one-line hints are at the end of this part.

EXERCISES 5

- 5.1 How many 5-digit numbers have their digits in strictly increasing order?
- 5.2 How many subsets of $\{1, 2, \dots, 10\}$, including the empty set, contain no two consecutive numbers?
- 5.3 In how many ways can a staircase of 10 steps be climbed taking 1, 2 or 3 steps at a time?
- 5.4 In how many ways can 6 rooks be placed on a 6×6 board, no two attacking each other, with no rook on the main diagonal?
- 5.5 The cells of a 3×3 table are filled with $1, 2, \dots, 9$. The median of each row is circled, and the median of the three circled numbers is 5. If N is the number of such tables, find the remainder when N is divided by 100. (*Hong Kong Preliminary 2018*)
- 5.6 A non-empty subset of $\{1, 2, \dots, 10\}$ is *good* if it has at least as many even numbers as odd numbers. If N is the number of good subsets, find $N/7$. (*Singapore 2010*)
- 5.7 In how many ways can A, B, C, D, E, F be placed at the vertices of a hexagon so that none of AB, CD, EF is a side, two placements being the same if one is a rotation of the other? (*Crux Mathematicorum*)
- 5.8 How many triangles with vertices among those of a regular 12-gon have no side in common with the 12-gon?
- 5.9 The diagonals of a convex hexagon are drawn, and no three meet at a point. Into how many regions do they cut the hexagon?
- 5.10 How many squares of all sizes are there on an 8×8 chessboard?
- 5.11 A graph on 7 vertices has no triangle. Find the largest possible number of edges.
- 5.12 The points of a 4×4 grid are 1 cm apart. How many circles of radius 1 cm pass through exactly two of the 16 points? (*Math Prize for Girls 2009*)
- 5.13 At a party of 12 people, each person shakes hands with exactly 5 others. How many handshakes are there?

PART II ANSWERS TO THE EXERCISES

Every answer below was checked by computer, by listing all the cases or by exact calculation. The hint gives only the first idea of the solution; if it is enough, complete the question yourself before reading further.

CHAPTER 1 · COUNTING PRINCIPLES

- 1.1 524 $x, y > 0: \sum_{x=1}^{35} \lfloor 35/x \rfloor = 131$; four sign choices.
- 1.2 2296 last digit 0: $9 \cdot 8 \cdot 7$; last digit 2, 4, 6, 8: $4 \cdot 8 \cdot 8 \cdot 7$.
- 1.3 1800 $\frac{7!}{2!} - 6! = 2520 - 720$.
- 1.4 66660 each digit 6 times in each place: $6 \cdot 10 \cdot 1111$.
- 1.5 3600 $7! - 2 \cdot 6!$.
- 1.6 60 $\frac{5!}{2}$.
- 1.7 570 three even, or one even and two odd: $\binom{10}{3} + 10\binom{10}{2}$.
- 1.8 330 hockey stick: $\binom{11}{4}$.
- 1.9 186 $\binom{10}{5} - \binom{6}{5} - 4\binom{6}{4} = 252 - 6 - 60$.
- 1.10 1296 two of the 9 horizontal and two of the 9 vertical lines: 36^2 .
- 1.11 43 a, b of opposite signs: 90 triples. A line comes from 2 of them, but $x - y = 0$ from 6: $\frac{84}{2} + 1$.
- 1.12 704 $a(b+1) + b(a+1) = 337$ gives $(2a+1)(2b+1) =$
- 675.
- 1.13 $5 \times 22, 8 \times 16$ $MN - \binom{M}{2} = 100$, so $M(2N - M + 1) = 200$.

CHAPTER 2 · DISTRIBUTIONS AND BIJECTIONS

- 2.1 816 $\binom{18}{3}$.
- 2.2 165 $\binom{11}{3}$.
- 2.3 286 add a slack variable: $\binom{13}{3}$.
- 2.4 125 $\binom{15}{3} - 4\binom{9}{3} + 6\binom{3}{3} = 455 - 336 + 6$.
- 2.5 219 first digit $a \geq 1$: $\binom{12}{3}$, minus the one case $a = 10$.
- 2.6 540 $3^6 - 3 \cdot 2^6 + 3$.
- 2.7 90 $540/3!$.
- 2.8 132 $\binom{10}{5} - \binom{4}{2}\binom{6}{3} = 252 - 120$.
- 2.9 132 C_6 .
- 2.10 495 $\binom{15-4+1}{4} = \binom{12}{4}$.
- 2.11 34 last part 2, or last part made 1 smaller: $a_n = a_{n-2} + a_{n-1}, a_2 = a_3 = 1$.
- 2.12 14 as many as partitions into parts 1, 2, 3: $a + 2b + 3c = 10$.

2.13 512 2^9 .

CHAPTER 3 · INCLUSION, EXCLUSION AND RECURSION

3.1 8 with $a = |A|, b = |B|, c = |A \cap B|, 2^a + 2^b - 2^c = 144$; the odd factor $9 = 8 + 2 - 1$ gives $c = 4, a = 7, b = 5$.

3.2 534 by 2 or 3: 667; of these, by 10 or 15: $100 + 66 - 33 = 133$.

3.3 265 D_6 .

3.4 315 $\binom{7}{3}D_4 = 35 \cdot 9$.

3.5 150 $3^5 - 3 \cdot 2^5 + 3$.

3.6 74460 $10^6 - 3 \cdot 9^6 + 3 \cdot 8^6 - 7^6$.

3.7 89 $t_n = t_{n-1} + t_{n-2}$ up to t_{10} .

3.8 144 $f_n = f_{n-1} + f_{n-2}$ with $f_1 = 2, f_2 = 3$.

3.9 57 second letter B: 8 choices for the third; otherwise $7 \cdot 7$.

3.10 70 list the losers in order and add the unused players of the winning team at the end: an order of 4 A's and 4 B's, $\binom{8}{4}$.

3.11 732 $3^6 + 3$.

3.12 56 $\frac{10^2 + 10 + 2}{2}$.

3.13 144 $\text{top} = a_1 + 5a_2 + 10a_3 + 10a_4 + 5a_5 + a_6$, so $5 \mid a_1 + a_6$; $6 \cdot 4!$.

CHAPTER 4 · PIGEONHOLE AND INVARIANTS

4.1 7 6 socks can be two of each colour.

4.2 51 $51, \dots, 100$ has no such pair; box each number by its largest odd divisor.

4.3 5 four parity boxes; $(0,0), (0,1), (1,0), (1,1)$ fail.

4.4 10 $3, 2, 1, 6, 5, 4, 9, 8, 7$ fails; $10 = 3^2 + 1$.

4.5 11 a rectangle's diagonals are diameters; 11 points fill two of the 9 diameter pairs, 10 can fill only one.

4.6 81 chains $x, 24x, 576x$ with $24 \nmid x$: 78 of length 2 and 3 of length 3, one removal each.

4.7 43 the triples $k, 89 - k, k(89 - k)$ for $k = 2, \dots, 44$ are disjoint; delete $2, \dots, 44$.

4.8 34 for $k = 33$, ages $2, 4, \dots, 14$ and $13, 15$ need the nine even cards; then $7, 9, 11$ take $21, 27, 33$ and 3 has none.

4.9 30 only 30 dark squares remain, and 30 dominoes fit.

4.10 4 a knight on a 4-row board attacks at most 6 squares; the 4 central squares work.

4.11 191 each move lowers the sum by 1: $210 - 19$.

4.12 16 sixteen 2×2 blocks, at most one king in each.

4.13 33 chains $x, 2x, 4x, \dots$ from odd x ; remove half of each chain, rounded down.

CHAPTER 5 · GRAPHS AND PROBLEM SOLVING

5.1 126 choose 5 of the digits $1, \dots, 9$: $\binom{9}{5}$.

5.2 144 $s_n = s_{n-1} + s_{n-2}$ with $s_1 = 2, s_2 = 3$.

5.3 274 split on the last move: $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, from $1, 2, 4$.

5.4 265 a derangement of 6: D_6 .

5.5 60 5 is the median of its row: $3 \cdot 16 \cdot 3!$; the other rows: $2 \cdot 10 \cdot (3!)^2$. $N = 207360$.

5.6 91 $N = 637$; count by the number of even elements.

5.7 32 fix A ; place B not next to it, then check CD and EF .

5.8 112 $\frac{n(n-4)(n-5)}{6}$ with $n = 12$.

5.9 25 $\binom{6}{4} + \binom{5}{2}$.

5.10 204 $1^2 + 2^2 + \dots + 8^2$.

5.11 12 Mantel: $\lfloor 49/4 \rfloor$; groups of 3 and 4.

5.12 52 two centres for each pair of points at most 2 apart; drop circles through three or more points.

5.13 30 $\frac{12 \cdot 5}{2}$.

[A NOTE BEFORE YOU CLOSE IT]

Four things worth remembering

Every IOQM answer is an integer from 00 to 99, so it is tempting to guess. Do not guess. A count is right only when nothing is counted twice and nothing is left out, so test the method on the smallest case you can list by hand. If it fails there, it fails everywhere.

Decide the method before you start. The table in section 5.6 sums up this part in eight rows: arrangements, selections, identical objects in boxes, grid paths, “none of these”, conditions on neighbours, “at least” or “always”, moves and tilings. If you cannot say which row a question belongs to, read it again before calculating.

Revise this part three times. The first pass, a few weeks out, is slow and complete, with every exercise attempted. The second, a week out, is only the gold cards and the worked examples. The third, on the morning of the paper, is only the dark cards and the common mistakes of section 5.7, since these are the mistakes most likely to happen in the paper.

Finally, tell us when we are wrong. If a line here does not match what you work out, write to ritesh@iitianforum.com or academics@iitianforum.com with the page and the line. Corrections go into the next edition, and the student who finds one is credited in it.

IOQM HANDBOOK RELEASE

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STATUS PART II · EDITION 1

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