

[OLYMPIAD MATHEMATICS]

IOQM · PRE-RMO

2027

IOQM *Handbook*

CLASS 8 TO 12

OLYMPIAD

THEORY AND PROBLEMS

IN THIS HANDBOOK

PAPER	IOQM MATHEMATICS
THIS VOLUME	PART III · GEOMETRY
EACH CHAPTER	THEORY · EXAMPLES · EXERCISES

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IOQM Mathematics Handbook

Part III · Geometry

FOR IOQM 2027 | EDITION 1 | 2026

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[IOQM · PART III]

Preface

IOQM is the first stage of the Indian mathematical olympiad programme, and for most students it is the first paper in which a formula is not enough. The questions are short and every answer is a whole number from 00 to 99, but almost every question needs an idea. This handbook is written to build those ideas, one chapter at a time.

The handbook follows the paper. It has four parts, one each for number theory, combinatorics, geometry and algebra, and every part is tied to the IOQM 2026 paper, so that you can see where each idea was actually tested. This volume is Part III, Geometry. Seven questions of IOQM 2026 were geometry questions, and most of them were opened by one good line in the figure: a parallelogram, a median, a circle that is not drawn.

Everything here stays within school mathematics. Nothing needs more than the geometry and trigonometry of Classes 9 to 11, and every result that is used is either proved in the text or stated with its conditions. The examples and exercises come from the IOQM 2026 paper, from olympiads of many countries, and from our own classes. Every solution has been written afresh, in the way we teach it, and every answer has been checked by computer, by exact calculation or with coordinates.

This book is meant to be worked through, not only read. Keep a notebook beside it, attempt every example before you read its solution, and come back later to the questions you could not do.

If something is not clear, if you find a mistake, or if you have a suggestion, write to me at ritesh@iitianforum.com or to our academic team at academics@iitianforum.com. Your message will help every student who uses this book after you.

Ritesh Raj

Founder, IITian Forum · IIT Delhi

[IOQM · PART III]

How to use this handbook

This page is for you, the student. Part III has five chapters, in the order in which they are taught. Chapter 1 covers angles, triangles and congruence. Chapter 2 is about similarity, lengths and areas, and Chapter 3 about trigonometry in geometry. Chapter 4 covers circles. Chapter 5 covers coordinates, shows how to choose the right method, and ends with a mixed problem set. The answers to all exercises, each with a one-line hint, are at the end of the part. This part can be read on its own.

HOW TO STUDY A CHAPTER

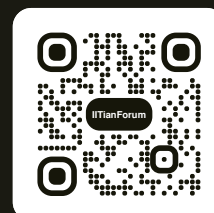
- 01** Read the theory slowly, once. Write out each short proof in your own notebook; that is how a result becomes yours.
- 02** Before reading a worked example, cover its solution, draw the figure large and to scale, and try the question for five minutes. Then compare.
- 03** Learn the gold cards by heart. They hold the definitions, theorems and key ideas that the paper uses again and again.
- 04** Read the dark cards twice. Each one marks a point where students often lose marks.
- 05** Attempt every exercise before you look at the answers. Read the one-line hint only when you are stuck, then finish the question yourself.
- 06** Keep a list of the questions you got wrong, and try them again a week later without looking at the solution.

Questions from IOQM 2026 are marked with their question number, and questions from other olympiads with their source. Stuck on a question, or found a mistake? Write to ritesh@iitianforum.com or academics@iitianforum.com.

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SCAN FOR THE COURSE

[IOQM · PART III]

Notation and conventions

Each symbol is defined, or first used, in the section shown.

SYMBOL	MEANING	SECTION
$\angle ABC$	the angle at B between BA and BC	1.2
AB	the segment AB , or its length	1.2
$\parallel, \perp$	is parallel to; is perpendicular to	1.1, 2.4
$\triangle ABC \cong \triangle PQR$	congruent, with A, B, C matching P, Q, R	1.3
a, b, c	the sides BC, CA, AB , opposite the angles A, B, C	1.4
$\triangle ABC \sim \triangle PQR$	similar, with the matching vertices in the same order	2.1
m_a	the median to the side a	2.4
$[ABC], \Delta$	the area of ABC ; the area of the triangle in question	2.3, 2.7
s	half the perimeter, $\frac{a+b+c}{2}$	2.5
h_a	the altitude to the side a	2.7
R, r	the circumradius; the inradius	3.2, 3.4
O, G, H, I	circumcentre, centroid, orthocentre, incentre	4.7
$\gcd(a, b)$	the greatest common divisor of a and b	5.2
$\iff$	"exactly when" (if and only if)	2.1

CONVENTIONS

- (i) **Figures.** A figure in a question is not drawn to scale unless the question says so. Draw your own, large and accurate.
- (ii) **Angles.** Angles are in degrees. "Angle A " of a triangle is $\angle BAC$.
- (iii) **Trapezium.** A quadrilateral with one pair of parallel sides.

- (iv) **Exact values.** Keep $\sqrt{2}, \sqrt{3}, \sqrt{5}$ and π exact until the last line.
- (v) **Numbering.** 3.5 is section 5 of Chapter 3. Examples are numbered in the same way, and Exercise 2.5 is question 5 of Exercises 2.
- (vi) **Sources.** "(IOQM 2026, Q24)" is question 24 of the IOQM 2026 paper. The name of another olympiad, as in "(RMO 2012)", gives the source of that question.

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CHAPTER 01 ANGLES, TRIANGLES AND CONGRUENCE

Most geometry questions in the paper ask for an angle, a length, an area or a ratio. Many of them need only the facts of this chapter: the angles made by parallel lines, the angle sums of triangles and polygons, isosceles triangles, congruence, the triangle inequality, Pythagoras' theorem and midpoints. Draw a large, accurate figure for every question; it often shows which of these facts to use.

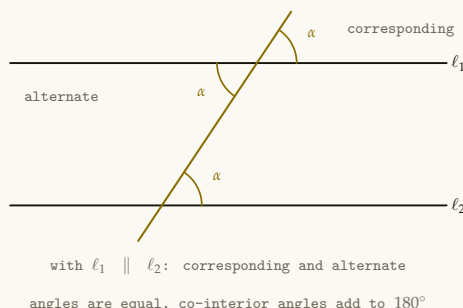
1.1 ANGLES, PARALLELS AND POLYGONS

THEOREM · ANGLE FACTS

Angles on a straight line add to 180° . Vertically opposite angles are equal.

If a line crosses two parallel lines, corresponding angles are equal, alternate angles are equal, and co-interior angles add to 180° .

The angles of a triangle add to 180° . The interior angles of a convex n -gon add to $(n-2) \cdot 180^\circ$, and its exterior angles add to 360° .



A regular n -gon has equal exterior angles, each $\frac{360^\circ}{n}$. For example, if each interior angle is 156° , each exterior angle is 24° , so $n = \frac{360}{24} = 15$.

An **equiangular hexagon** has all six angles equal to 120° . Extend three alternate sides. Each corner piece has two angles of 60° , so it is an equilateral triangle, and the three extended sides form a large equilateral triangle.

EXAMPLE 1.1

An equiangular hexagon has sides 4, 8, 10, 14, 15 and 19 in some order. It is an equilateral triangle of side 37 with three equilateral corners cut off. The same hexagon is also an equilateral triangle of side n with three corners cut off, where $n \neq 37$. Find n . (British Juniors 1999)

Let the sides be $a_1, a_2, \dots, a_6$ in order. Their sum is 70.

In the triangle of side 37, let the cut edges be a_1, a_3, a_5 . Each side of the triangle is a side of the hexagon with a corner piece at each end:

$$37 = a_1 + a_2 + a_3 = a_3 + a_4 + a_5 = a_5 + a_6 + a_1$$

On adding,

$$111 = 70 + (a_1 + a_3 + a_5)$$

as each of a_1, a_3, a_5 appears twice. $\therefore a_1 + a_3 + a_5 = 41$ and $a_2 + a_4 + a_6 = 29$.

The other triangle comes from extending a_1, a_3, a_5 , so its corner pieces are a_2, a_4, a_6 . Similarly,

$$3n = 70 + 29 = 99$$

One order that works is 14, 15, 8, 10, 19, 4.

ANSWER $n = 33$

1.2 ISOSCELES TRIANGLES AND THE EXTERIOR ANGLE

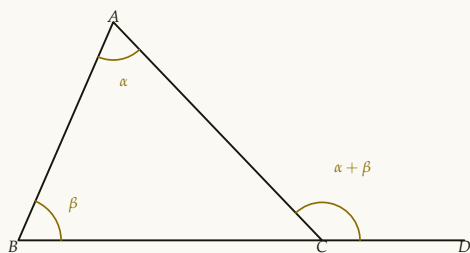
THEOREM · TRIANGLE FACTS

The exterior angle at a vertex equals the sum of the interior angles at the other two vertices.

Two sides of a triangle are equal exactly when the angles opposite them are equal. The longer side is opposite the larger angle.

An equilateral triangle has all three angles equal to 60° .

When a figure has several isosceles triangles, call one angle x and go round the figure: write each angle in terms of x , using the equal base angles and the exterior angle. The angle sum of one triangle then gives an equation for x .



the exterior angle at C equals the sum
of the two interior angles opposite it

KEY IDEA · BUILD THE POINT YOU EXPECT

If a given point is hard to reach, construct a new point with a property you can use. Then show that the new point has the given property too, so the two points are the same.

EXAMPLE 1.2

Triangle ABC is equilateral. The ray BE meets the side AC at D , with $\angle CBE = 20^\circ$ and $DE = AB$. Find $\angle BEC$. (Belgium Geometry Olympiad 2007)

Let $E' \neq B$ be the point on the ray BD with $CE' = CB$. We show that $E' = E$.

$CE' = CB$, so in triangle BCE' , $\angle CE'B = \angle CBE' = 20^\circ$ and $\angle BCE' = 180^\circ - 40^\circ = 140^\circ$. $\therefore \angle DCE' = 140^\circ - 60^\circ = 80^\circ$.

In triangle BDC , $\angle BDC = 180^\circ - 20^\circ - 60^\circ = 100^\circ$, so $\angle CDE' = 180^\circ - 100^\circ = 80^\circ$.

In triangle CDE' , $\angle DCE' = \angle CDE' = 80^\circ$, so $DE' = CE'$.

Now $DE' = CE' = CB = AB = DE$. Also E lies beyond D , as $DE = AB > BD$. So E and E' lie on the same ray from D at the same distance, i.e., $E' = E$. Hence $\angle BEC = \angle CE'B = 20^\circ$.

ANSWER 20°

1.3 CONGRUENCE

THEOREM · CONGRUENCE TESTS

Two triangles are congruent if they agree in

SSS: all three sides;

SAS: two sides and the angle between them;

ASA or AAS: two angles and one side in the same position;

RHS: a right angle, the hypotenuse and one more side.

Two sides and an angle that is not between them is not a test: two different triangles can have them.

A common use: if the bisector of $\angle A$ meets a segment XY at right angles, with X on AB and Y on AC , the two right triangles it forms are congruent by ASA, so $AX = AY$.

EXAMPLE 1.3

The sides of a triangle are three consecutive integers. A median of the triangle is perpendicular to an angle bisector. Find the perimeter. (Moldova 1997)

A median and a bisector from the same vertex are never perpendicular. So let the bisector from A be perpendicular to the median BM , where M is the midpoint of AC .

By the common use above, with $X = B$ and $Y = M$,

$$AB = AM = \frac{1}{2}AC$$

So one side is twice another. With sides $n, n+1, n+2$, the three pairs give:

equation	n	sides
$2n = n+1$	1	1, 2, 3 (not a triangle)
$2n = n+2$	2	2, 3, 4
$2(n+1) = n+2$	0	none

Only 2, 3, 4 is left, with $AB = 2$, $AC = 4$, $BC = 3$. Check: $AB = AM = 2$, so triangle ABM is isosceles and the bisector from A is perpendicular to BM .

ANSWER $2+3+4=9$

WHERE MARKS GO · MATCH THE VERTICES

Write the matching vertices in the same order. Then the equal sides can be read from the letters: $\triangle ABC \cong \triangle PQR$ gives $AB = PQ$, $BC = QR$, $CA = RP$.

1.4 THE TRIANGLE INEQUALITY

THEOREM · WHEN THREE LENGTHS MAKE A TRIANGLE

Three lengths $a \leq b \leq c$ are the sides of a triangle exactly when $c < a + b$.

With c the longest side, the angle opposite c is acute, right or obtuse as c^2 is less than, equal to or greater than $a^2 + b^2$.

For example, 5, 7, 12 is not a triangle, since $12 = 5 + 7$.

The sides 5, 7, 8 give an acute triangle, since $64 < 25 + 49$.

EXAMPLE 1.4

For how many integers x are 7, 8 and x the sides of an obtuse triangle?

For a triangle, $8 - 7 < x < 8 + 7$, so $2 \leq x \leq 14$.

The largest angle is opposite the longest side. Taking cases on the longest side,

longest side	obtuse when	x
x ($x \geq 8$)	$x^2 > 49 + 64 = 113$	11, 12, 13, 14
8 ($x \leq 8$)	$64 > 49 + x^2$	2, 3

In the second row, $x^2 < 15$, so $x \leq 3$.

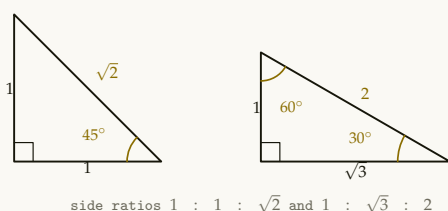
ANSWER $4 + 2 = 6$

1.5 PYTHAGORAS AND SPECIAL TRIANGLES**THEOREM · PYTHAGORAS**

If $\angle C = 90^\circ$, then $a^2 + b^2 = c^2$. Conversely, if $a^2 + b^2 = c^2$, then $\angle C = 90^\circ$.

The right isosceles triangle has sides in the ratio $1 : 1 : \sqrt{2}$. The triangle with angles $30^\circ, 60^\circ, 90^\circ$ has sides $1 : \sqrt{3} : 2$, with the shortest side opposite 30° .

The second triangle is half of an equilateral triangle, cut along an altitude. So an equilateral triangle of side s has height $\frac{\sqrt{3}}{2}s$ and area $\frac{\sqrt{3}}{4}s^2$. Learn the triples $(3, 4, 5)$, $(5, 12, 13)$, $(8, 15, 17)$, $(7, 24, 25)$ and $(20, 21, 29)$, and their multiples.

**EXAMPLE 1.5**

P is a point inside a rectangle $ABCD$ with $PA = 3$, $PB = 4$ and $PC = 5$. Find PD .

Draw the lines through P parallel to the sides. Let the distances of P from AD , BC , AB , CD be x , x' , y , y' .

Each of PA, PB, PC, PD is the hypotenuse of a right triangle, so

$$PA^2 = x^2 + y^2, \quad PB^2 = x'^2 + y^2$$

$$PC^2 = x'^2 + y'^2, \quad PD^2 = x^2 + y'^2$$

Both $PA^2 + PC^2$ and $PB^2 + PD^2$ equal $x^2 + x'^2 + y^2 + y'^2$, so

$$9 + 25 = 16 + PD^2, \quad PD^2 = 18$$

ANSWER $PD = 3\sqrt{2}$

The rule $PA^2 + PC^2 = PB^2 + PD^2$ holds for every point P and every rectangle $ABCD$. It is called the **British flag theorem**.

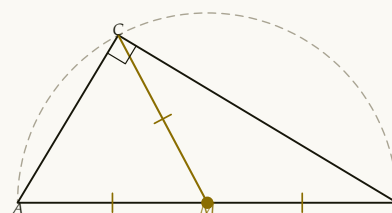
1.6 THE MEDIAN TO THE HYPOTENUSE**THEOREM · MEDIAN TO THE HYPOTENUSE**

In a right triangle, the midpoint M of the hypotenuse is at the same distance from all three vertices. It is the centre of the circle through them.

Conversely, if the median CM is half of AB , then $\angle C = 90^\circ$.

The median cuts the right triangle into two isosceles triangles. So a right angle gives pairs of equal angles, and that is how the theorem is used in angle questions.

For example, if $\angle C = 90^\circ$ and $\angle A = \alpha$, then $\angle MCA = \alpha$ and $\angle MCB = 90^\circ - \alpha$. In the triangle with sides 6, 8, 10, the median to the hypotenuse is 5.



$MA = MB = MC$: the midpoint of the hypotenuse is the circumcentre

EXAMPLE 1.6

Four points A, B, C, D lie on a line in this order. A point E , not on the line, satisfies $\angle AEB = \angle BEC = \angle CED = 45^\circ$. Let F and G be the midpoints of AC and BD . If $\angle FEG = x^\circ$, find x . (IOQM 2026, Q19)

$45^\circ + 45^\circ = 90^\circ$, so

$$\angle AEC = 90^\circ, \quad \angle BED = 90^\circ$$

F is the midpoint of the hypotenuse AC , so $FE = FA$. In isosceles triangle FAE , let $\angle AEF = \angle EAF = \alpha$.

Similarly, G is the midpoint of the hypotenuse BD , so $GE = GD$. In isosceles triangle GED , let $\angle GED = \angle EDG = \delta$.

In triangle AED , $\angle AED = 3 \cdot 45^\circ = 135^\circ$, so $\alpha + \delta = 45^\circ$. Hence

$$\angle FEG = 135^\circ - \alpha - \delta = 135^\circ - 45^\circ = 90^\circ$$

ANSWER 90

1.7 MIDPOINTS AND PARALLELOGRAMS

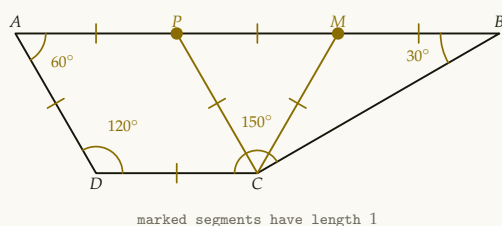
THEOREM · MIDPOINT THEOREM

If M and N are the midpoints of AB and AC , then $MN \parallel BC$ and $MN = \frac{1}{2}BC$.

In a parallelogram, opposite sides are equal and parallel, opposite angles are equal, and the diagonals bisect each other. A quadrilateral with one pair of opposite sides equal and parallel is a parallelogram.

In a trapezium, the segment joining the midpoints of the two legs is parallel to the bases and equal to half their sum.

In a trapezium, a segment from a vertex parallel to a leg cuts off a parallelogram. It moves that leg across, next to the other leg, where the two can be compared.



EXAMPLE 1.7

In trapezium $ABCD$, $AB \parallel CD$, $AB = 3CD$, $CD = DA$ and $\angle CDA = 120^\circ$. If the largest angle of $ABCD$ is x° and the smallest is y° , find x/y . (IOQM 2026, Q8)

The angles do not depend on the size, so let $CD = DA = 1$ and $AB = 3$.

$AB \parallel CD$, so $\angle A = 180^\circ - 120^\circ = 60^\circ$ (co-interior angles).

Take P on AB with $AP = 1$. Then $AP = DC$ and $AP \parallel DC$, so $APCD$ is a parallelogram, and $PC = AD = 1$, $PC \parallel AD$.

$\therefore \angle CPB = \angle DAB = 60^\circ$ (corresponding angles), and $PB = 3 - 1 = 2$.

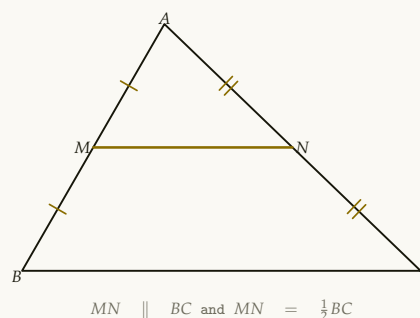
Let M be the midpoint of PB , so $PM = MB = 1$. In triangle PMC , $PM = PC = 1$ and $\angle MPC = 60^\circ$, so it is equilateral and $MC = 1$.

In triangle MBC , $MB = MC = 1$ and $\angle CMB = 180^\circ - 60^\circ = 120^\circ$, so $\angle B = 30^\circ$. Hence $\angle C = 180^\circ - 30^\circ = 150^\circ$.

The angles are $60^\circ, 30^\circ, 150^\circ, 120^\circ$, so

$$\frac{x}{y} = \frac{150}{30} = 5$$

ANSWER 5



Join the midpoints of the sides of any quadrilateral. Each new side is parallel to a diagonal and half as long, so the new figure is a parallelogram whose perimeter is the sum of the diagonals (**Varignon's theorem**).

WHERE MARKS GO · A FIGURE TO SCALE

Draw the figure to scale with a ruler and protractor before any working. In the last example, a measured angle of about 30° at B gives the answer at once, and it catches a slip in the working.

EXERCISES 1

- 1.1 Each interior angle of a regular polygon is 168° . How many sides does it have?
- 1.2 The interior angles of a convex polygon add up to 2340° . How many diagonals does it have?
- 1.3 An equiangular hexagon has four consecutive sides 1, 7, 2, 4, in this order. Find its perimeter.
- 1.4 In triangle ABC , $AB = AC$, and D is a point on AC with $BD = BC = AD$. Find $\angle A$.
- 1.5 E is a point inside the square $ABCD$ such that triangle ABE is equilateral. Find $\angle DEC$.
- 1.6 How many triangles with integer sides have perimeter 15? Congruent triangles are counted once.
- 1.7 For how many integers x are 10, 24 and x the sides of an acute triangle?
- 1.8 P is a point inside a rectangle $ABCD$ with $PA = 1$, $PB = 7$ and $PC = 8$. Find PD .
- 1.9 In triangle ABC , $\angle C = 90^\circ$, M is the midpoint of AB and $\angle MCB = 25^\circ$. Find $\angle A$.
- 1.10 The diagonals of a quadrilateral have lengths 10 and 14. Find the perimeter of the quadrilateral formed by the midpoints of its sides.
- 1.11 In triangle ABC , M is the midpoint of BC and N is the midpoint of AM . The line BN meets AC at P . Find $AP : PC$.
- 1.12 Equilateral triangles AEB , BFC , CGD and DHA are drawn outwards on the sides of the square $ABCD$. If the area of $EFGH$ is 25, find the area of $ABCD$. (*Crux Mathematicorum*, *Mathematic MA-207*)

CHAPTER 02 SIMILARITY, LENGTHS AND AREAS

A length or an area in a figure is usually found through a ratio. This chapter covers similar triangles, the altitude of a right triangle, the angle bisector theorem, Stewart's theorem and the median, areas and their ratios, the theorems of Ceva and Menelaus, and the link between the altitudes and the sides of a triangle.

2.1 SIMILAR TRIANGLES

THEOREM · SIMILARITY TESTS

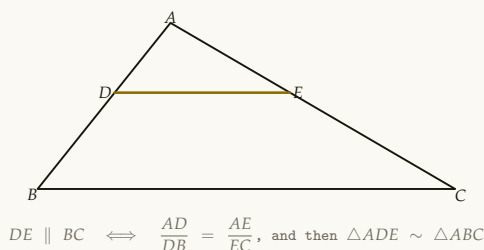
Two triangles are similar if

AA: two pairs of angles are equal;

SAS: two pairs of sides are in the same ratio, and the angles between them are equal;

SSS: all three pairs of sides are in the same ratio.

We write $\triangle ABC \sim \triangle PQR$, with the matching vertices in the same order. If lengths are in the ratio k , areas are in the ratio k^2 .



A line parallel to one side of a triangle cuts off a smaller triangle similar to it. So it divides the other two sides in the same ratio.

EXAMPLE 2.1

A triangle ABC is divided into four regions by three lines parallel to BC . The lines divide AB into four equal parts. If the second-largest region has area 25, find the area of triangle ABC . (Australian Intermediate Mathematics Olympiad 2017)

Let S be the area of ABC . Each line cuts off a triangle at A similar to ABC , with sides $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}$ of those of ABC , so their areas are $\frac{1}{16}S, \frac{4}{16}S, \frac{9}{16}S$.

Each region is the difference of two of these triangles. Numbering the regions from A ,

region	area
1	$\frac{1}{16}S$
2	$\frac{4-1}{16}S = \frac{3}{16}S$
3	$\frac{9-4}{16}S = \frac{5}{16}S$
4	$\frac{16-9}{16}S = \frac{7}{16}S$

The second-largest region is region 3, so $\frac{5}{16}S = 25$.

ANSWER $S = 80$

EXAMPLE 2.2

Triangle ABC has centroid G and $BC = 3$. Triangle ABC is similar to triangle GAB , with the vertices in this order. If the area of ABC is x , find $2x^2$. (CMIMC Team 2020)

Let M be the midpoint of BC . Then G lies on the median AM and $AG = \frac{2}{3}AM$.

$\triangle ABC \sim \triangle GAB$, so $\angle ABC = \angle GAB$, i.e., $\angle ABM = \angle BAM$. So $\triangle ABM$ is isosceles and $MA = MB = \frac{3}{2}$.

Now $MA = MB = MC$, so $\angle BAC = 90^\circ$ (Section 1.6). Also $AG = \frac{2}{3} \cdot \frac{3}{2} = 1$.

Again from the similarity, $\frac{BC}{AB} = \frac{AB}{GA}$, so $AB^2 = BC \cdot GA = 3$.

As $\angle BAC = 90^\circ$, $AC^2 = 9 - 3 = 6$ and

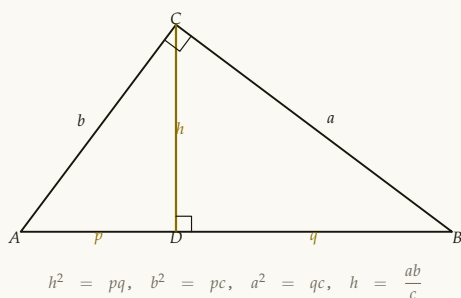
$$x = \frac{1}{2} \cdot AB \cdot AC = \frac{1}{2} \sqrt{18}, \quad x^2 = \frac{18}{4}$$

ANSWER $2x^2 = 9$

2.2 THE ALTITUDE OF A RIGHT TRIANGLE**THEOREM · ALTITUDE TO THE HYPOTENUSE**

In triangle ABC with $\angle C = 90^\circ$, let CD be the altitude, with $AD = p$, $DB = q$ and $CD = h$. The triangles ACD , CBD and ABC are similar, so

$$h^2 = pq, \quad b^2 = pc, \quad a^2 = qc, \quad h = \frac{ab}{c}.$$



For example, if $p = 9$ and $q = 16$, then $c = 25$, $h = \sqrt{144} = 12$, $b = \sqrt{225} = 15$ and $a = \sqrt{400} = 20$.

2.3 PARALLEL LINES AND RATIOS

Two parallel segments give a pair of similar triangles. Either the triangles are in an X shape, with the vertically opposite angles at the crossing point, or in an A

shape, one triangle inside the other.

For example, if the diagonals of a trapezium $ABCD$ with $AB \parallel CD$ meet at O , the triangles OAB and OCD form an X shape, so

$$OA : OC = OB : OD = AB : CD.$$

If $DE \parallel BC$ in triangle ABC , the triangles ADE and ABC form an A shape, so $AD : AB = AE : AC = DE : BC$.

EXAMPLE 2.3

$PBCQ$ is a trapezium with $PQ \parallel BC$ and $PQ : BC = 2 : 3$. Its diagonals PC and QB meet at A . If the area of triangle ABC is 36, find the area of $PBCQ$. (Crux Mathematicorum, Mathematic MA-171)

$PQ \parallel BC$, so $\triangle APQ \sim \triangle ACB$ (an X shape), with ratio $2 : 3$, so

$$AP : AC = 2 : 3, \quad [APQ] = \left(\frac{2}{3}\right)^2 \cdot 36 = 16$$

$\triangle ABP$ and $\triangle ABC$ have the same height from B , so their areas are in the ratio of their bases $AP : AC$:

$$[ABP] = \frac{2}{3} \cdot 36 = 24$$

Similarly, $[AQC] = 24$. Adding the four triangles,

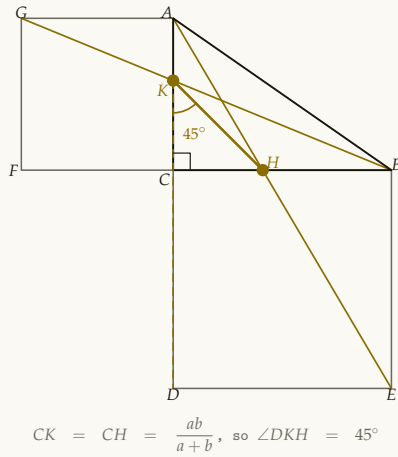
$$[PBCQ] = 36 + 16 + 24 + 24$$

ANSWER 100

KEY IDEA · LOOK FOR PARALLEL LINES

When two segments in a figure are parallel, find the two similar triangles they make. The ratio of the parallel segments then gives the other ratios.

Similar triangles also come without parallel lines. If D is on AB and E is on AC with $\angle ADE = \angle ACB$, the triangles ADE and ACB are similar (AA), so $AD \cdot AB = AE \cdot AC$.

**EXAMPLE 2.4**

Triangle ABC has a right angle at C . Squares $BCDE$ and $ACFG$ are drawn outwards on BC and CA . AE meets BC at H , and BG meets AC at K . Find $\angle DKH$. (Australian Mathematical Olympiad 2005)

Let $BC = a$ and $CA = b$.

$\angle ACB = \angle BCD = 90^\circ$, so D lies on the line AC , beyond C .

$AC \parallel BE$ (both are perpendicular to BC), so $\triangle HCA \sim \triangle HBE$:

$$\frac{CH}{HB} = \frac{CA}{BE} = \frac{b}{a}$$

As $CH + HB = a$, we get $CH = \frac{ab}{a+b}$.

Similarly, $BC \parallel AG$ gives $\frac{CK}{KA} = \frac{a}{b}$, so $CK = \frac{ab}{a+b}$.

$\therefore CK = CH$, and $\triangle KCH$ is right isosceles, so $\angle CKH = 45^\circ$.

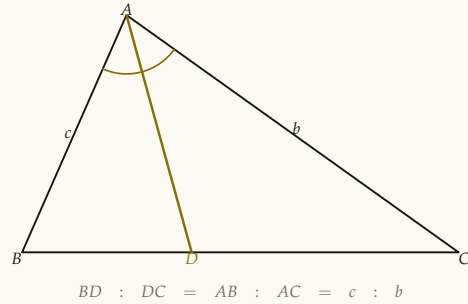
D lies on the ray KC , so $\angle DKH = \angle CKH$.

ANSWER 45°
2.4 BISECTORS, STEWART AND MEDIANS
THEOREM · ANGLE BISECTOR THEOREM

If AD bisects $\angle A$, with D on BC , then $BD : DC = AB : AC = c : b$. So

$$BD = \frac{ac}{b+c}, \quad DC = \frac{ab}{b+c}.$$

If $AB \neq AC$, the external bisector at A meets the line BC outside the segment, in the same ratio.

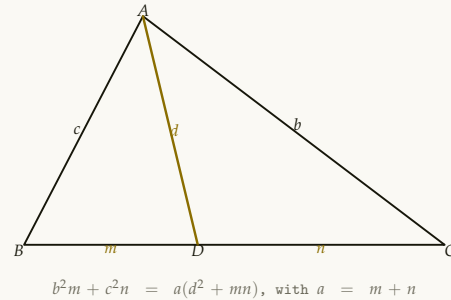

THEOREM · STEWART'S THEOREM

If D is on BC with $BD = m$, $DC = n$, $AD = d$ and $a = m + n$, then

$$b^2m + c^2n = a(d^2 + mn).$$

Median ($m = n$): $b^2 + c^2 = 2m_a^2 + \frac{1}{2}a^2$.

Bisector: $d^2 = bc - mn$.



In Stewart's theorem each side from A is multiplied by the far part of BC : b ends at C and goes with $m = BD$. The median form is **Apollonius' theorem**. It holds for any point E and any segment AC with midpoint O : $EA^2 + EC^2 = 2EO^2 + 2OA^2$.

EXAMPLE 2.5

In triangle ABC , the point D lies on AB with $AD = 25$ and $BD = 24$. Also $BC = 28$ and $CD = 20$. Find AC . (Malaysia 2019)

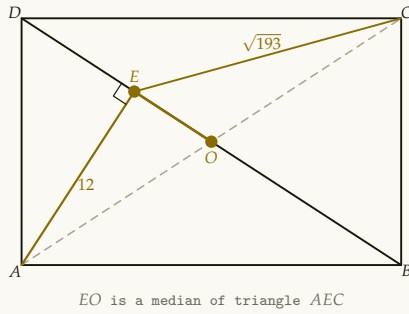
Here $AB = AD + DB = 49$. By Stewart's theorem in $\triangle ABC$ with the segment CD (the side AC goes with the far part DB , and BC with AD),

$$AC^2 \cdot 24 + BC^2 \cdot 25 = AB(CD^2 + AD \cdot DB)$$

$$24 AC^2 + 784 \cdot 25 = 49(400 + 600)$$

$$24 AC^2 + 19600 = 49000, \quad AC^2 = 1225$$

ANSWER $AC = 35$

**EXAMPLE 2.6**

Let $ABCD$ be a rectangle, and let E be the point on BD with $AE \perp BD$. If $AE = 12$ and $CE = \sqrt{193}$, find the area of $ABCD$. (IOQM 2026, Q24)

Let O be the centre of the rectangle and $d = AC = BD$. O is the midpoint of both diagonals, so $OA = OC = \frac{d}{2}$.

E and O lie on BD , and $AE \perp BD$, so in the right triangle AEO ,

$$OE^2 = OA^2 - AE^2 = \frac{d^2}{4} - 144$$

In $\triangle AEC$, EO is a median, so by Apollonius' theorem,

$$EA^2 + EC^2 = 2EO^2 + 2OA^2$$

$$144 + 193 = 2\left(\frac{d^2}{4} - 144\right) + 2 \cdot \frac{d^2}{4} = d^2 - 288$$

$\therefore d^2 = 625$ and $d = 25$.

The rectangle is twice $\triangle ABD$, whose base is BD and height is AE :

$$[ABCD] = 2 \cdot \frac{1}{2} \cdot BD \cdot AE = 25 \cdot 12$$

ANSWER 300**2.5 AREAS AND THEIR RATIOS****THEOREM · AREA FACTS**

$$[ABC] = \frac{1}{2} \text{ base} \cdot \text{height} = \frac{1}{2} ab \sin C.$$

Triangles with the same height have areas in the ratio of their bases.

If D is on AB and E is on AC , then

$$\frac{[ADE]}{[ABC]} = \frac{AD}{AB} \cdot \frac{AE}{AC}.$$

Heron: with $s = \frac{a+b+c}{2}$,

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)}.$$

For example, the triangle with sides 13, 14, 15 has $s = 21$ and area $\sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$.

EXAMPLE 2.7

Triangle ABC has area 300. Q is the midpoint of BC , P is on AC with $CP = 3PA$, and R is on AB with $[PQR] = 2[RBQ]$. Find $[PQR]$. (Australian Intermediate Mathematics Olympiad 2018)

Let $S = 300$ and $AR = t \cdot AB$, so $RB = (1-t)AB$.

Each of the three corner triangles has an angle common with $\triangle ABC$, so by the third area fact,

triangle	ratios	area
APR	$\frac{1}{4} \cdot t$	$\frac{t}{4}S$
RBQ	$(1-t) \cdot \frac{1}{2}$	$\frac{1-t}{2}S$
QCP	$\frac{1}{2} \cdot \frac{3}{4}$	$\frac{3}{8}S$

What is left of $\triangle ABC$ is $\triangle PQR$:

$$[PQR] = S \left(1 - \frac{t}{4} - \frac{1-t}{2} - \frac{3}{8} \right) = S \left(\frac{1}{8} + \frac{t}{4} \right)$$

As $[PQR] = 2[RBQ]$, we get $\frac{1}{8} + \frac{t}{4} = 1 - t$, so $t = \frac{7}{10}$ and

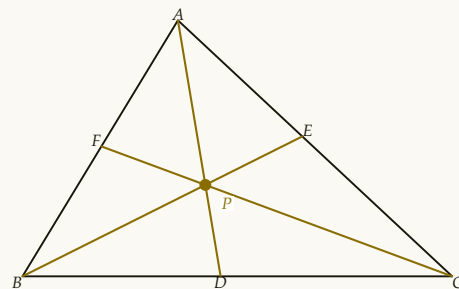
$$[PQR] = S \left(\frac{1}{8} + \frac{7}{40} \right) = \frac{3}{10} \cdot 300$$

ANSWER 90**2.6 CEVA AND MENELAUS****THEOREM · CEVA AND MENELAUS**

Ceva: with D, E, F on the sides BC, CA, AB , the lines AD, BE, CF meet at one point exactly when

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

Menelaus: if a straight line meets the lines BC, CA, AB at D, E, F , the same product equals 1.



concurrent exactly when $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$

To write the product, go round the triangle: vertex, point, vertex, point, vertex, point.

For the medians every ratio is 1, so the medians meet at one point, the centroid. For the angle bisectors the bisector theorem gives the product $\frac{c}{b} \cdot \frac{a}{c} \cdot \frac{b}{a} = 1$, so they also meet at one point, the incentre.

EXAMPLE 2.8

In triangle ABC , D and E are points on BC with $BD = DE = EC$, and F is the midpoint of AC . BF meets AD at P and AE at Q . If $[APQ] = 72$, find the area of $DEQP$. (RMO 2012)

The line BF meets the sides of $\triangle ADC$ at P (on AD), B (on CD produced) and F (on CA). By Menelaus' theorem, going A, P, D, B, C, F ,

$$\frac{AP}{PD} \cdot \frac{DB}{BC} \cdot \frac{CF}{FA} = 1, \quad \frac{AP}{PD} \cdot \frac{1}{3} \cdot 1 = 1$$

$$\therefore AP : PD = 3 : 1.$$

Similarly, the same line meets the sides of $\triangle AEC$ at Q, B, F :

$$\frac{AQ}{QE} \cdot \frac{EB}{BC} \cdot \frac{CF}{FA} = 1, \quad \frac{AQ}{QE} \cdot \frac{2}{3} \cdot 1 = 1$$

$$\therefore AQ : QE = 3 : 2.$$

$\triangle APQ$ and $\triangle ADE$ have the same angle at A , so

$$\frac{[APQ]}{[ADE]} = \frac{3}{4} \cdot \frac{3}{5} = \frac{9}{20}$$

$$\text{So } [ADE] = 72 \cdot \frac{20}{9} = 160, \text{ and } [DEQP] = 160 - 72.$$

ANSWER 88**2.7 ALTITUDES AND SIDES**

Twice the area of a triangle is $a h_a = b h_b = c h_c$. So each side is 2Δ divided by its altitude, where Δ is the area: the longest side has the shortest altitude.

EXAMPLE 2.9

In triangle ABC , the altitudes AD and BE have lengths 40 and 120. Find the largest possible integer length of the third altitude. (Singapore 2009)

Let h be the third altitude. The sides are

$$a = \frac{2\Delta}{40}, \quad b = \frac{2\Delta}{120}, \quad c = \frac{2\Delta}{h}$$

So $\frac{1}{40}, \frac{1}{120}, \frac{1}{h}$ satisfy the triangle inequality (divide it by 2Δ):

$$\frac{1}{h} < \frac{1}{40} + \frac{1}{120} = \frac{1}{30}, \quad \text{so } h > 30;$$

$$\frac{1}{h} > \frac{1}{40} - \frac{1}{120} = \frac{1}{60}, \quad \text{so } h < 60$$

$h = 59$ works: as $30 < 59 < 60$, there is a triangle with sides $\frac{1}{40}, \frac{1}{120}, \frac{1}{59}$. Its altitudes are in the ratio $40 : 120 : 59$, and an enlargement makes them 40, 120, 59.

ANSWER 59**EXAMPLE 2.10**

A triangle has sides x, y and 20, where $x > y > 20$ are integers. Its altitudes to these sides are h_x, h_y, h_{20} . If $h_{20} = h_x + h_y$, find the perimeter. (AIMO 2022)

Each altitude is 2Δ divided by its side. On cancelling 2Δ ,

$$\frac{1}{20} = \frac{1}{x} + \frac{1}{y}$$

On multiplying by $20xy$,

$$xy = 20x + 20y, \quad (x - 20)(y - 20) = 400$$

$x > y > 20$, so $x - 20 > y - 20 > 0$. Factor pairs of 400, with the triangle inequality $y + 20 > x$ (x is the longest side):

$x - 20, y - 20$	x, y	$y + 20 > x?$
400, 1	420, 21	no
200, 2	220, 22	no
100, 4	120, 24	no
80, 5	100, 25	no
50, 8	70, 28	no
40, 10	60, 30	no
25, 16	45, 36	yes

ANSWER 45 + 36 + 20 = 101**WHERE MARKS GO · CHECK THE TRIANGLE EXISTS**

When a question gives lengths by a condition, test the triangle inequality on the answer. In the last example six of the seven factor pairs fail it.

EXERCISES 2

- 2.1 A square is drawn inside a right triangle with legs 6 and 3, with two of its sides along the legs and one corner on the hypotenuse. Find its side.
- 2.2 In triangle ABC , D is on AB and E is on AC with $DE \parallel BC$ and $AD : DB = 2 : 3$. If $[ABC] = 75$, find the area of $DBCE$.
- 2.3 The altitude to the hypotenuse of a right triangle divides the hypotenuse into parts 4 and 9. Find the area of the triangle.
- 2.4 $ABCD$ is a parallelogram of area 60 and E is the midpoint of CD . BE meets AC at F . Find the area of triangle CEF .
- 2.5 In triangle ABC , $AB = 6$, $AC = 9$ and $BC = 10$. The bisector of $\angle A$ meets BC at D . Find AD^2 .
- 2.6 Find the length of the median to the side 8 in the triangle with sides 7, 8, 9.
- 2.7 Find the length of the altitude to the side 14 in the triangle with sides 13, 14, 15.
- 2.8 A triangle has area 114 and sides of integer length. What is its perimeter? (*Math Prize for Girls 2014*)
- 2.9 In triangle ABC , the lines AD , BE , CF meet at one point, with $BD : DC = 2 : 3$ and $CE : EA = 1 : 2$. Find $AF : FB$.
- 2.10 The sides of triangle ABC are $AB = 104$, $BC = 112$, $CA = 120$. D and F are the midpoints of BC and AB . The perpendicular from F to AC and the perpendicular to BC at B meet at N . Find ND . (*RMO 2008*)
- 2.11 $ABCDEF$ is a regular hexagon and P is the midpoint of AB . The segment FC meets PE at Q . Find the ratio $[DEQ] : [FPQ]$. (*Crux Mathematicorum, Mathematic*)

CHAPTER 03 TRIGONOMETRY IN GEOMETRY

When an angle chase gets stuck, trigonometry turns the figure into equations. This chapter covers the values of the special angles, the sine rule, the cosine rule, areas with sines, and identities that use $A + B + C = 180^\circ$. Keep every value exact: write $\sqrt{3}$ and $\sqrt{5}$, never a decimal.

3.1 VALUES AND FORMULAS

RULES · VALUES TO KNOW

$$\sin 15^\circ = \cos 75^\circ = (\sqrt{6} - \sqrt{2})/4$$

$$\cos 15^\circ = \sin 75^\circ = (\sqrt{6} + \sqrt{2})/4$$

$$\tan 15^\circ = 2 - \sqrt{3}, \quad \tan 75^\circ = 2 + \sqrt{3}$$

$$\sin 18^\circ = \cos 72^\circ = (\sqrt{5} - 1)/4$$

$$\cos 36^\circ = \sin 54^\circ = (\sqrt{5} + 1)/4$$

For example, $\tan 75^\circ = \tan(45^\circ + 30^\circ) = 2 + \sqrt{3}$, and then

$$\tan 15^\circ = \frac{1}{\tan 75^\circ} = \frac{1}{2 + \sqrt{3}} = 2 - \sqrt{3}.$$

EXAMPLE 3.1

In an isosceles triangle ABC with $\angle ACB = 90^\circ$, the point D is on the side BC with $\angle ADC = 75^\circ$. If the area of triangle ADC is 81, find BD . (*IOQM 2026, Q13*)

Let $CA = CB = a$. In triangle ACD , $\angle C = 90^\circ$, so

$$CD = \frac{AC}{\tan 75^\circ} = a \tan 15^\circ = a(2 - \sqrt{3})$$

The area of triangle ADC is 81, so

$$\frac{1}{2} \cdot a \cdot a(2 - \sqrt{3}) = 81$$

$$a^2 = \frac{162}{2 - \sqrt{3}} = 162(2 + \sqrt{3})$$

Now $BD = a - a(2 - \sqrt{3}) = a(\sqrt{3} - 1)$, so

$$\begin{aligned} BD^2 &= 162(2 + \sqrt{3})(4 - 2\sqrt{3}) \\ &= 324(2 + \sqrt{3})(2 - \sqrt{3}) = 324 \end{aligned}$$

ANSWER $BD = 18$

RULES · FORMULAS

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

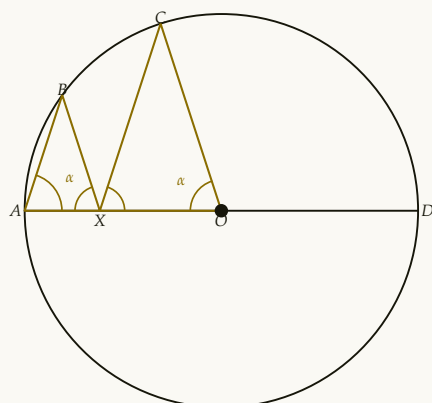
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = 1 - 2 \sin^2 x = 2 \cos^2 x - 1$$

$$2 \cos x \cos y = \cos(x + y) + \cos(x - y)$$

$$2 \sin x \cos y = \sin(x + y) + \sin(x - y)$$

all four marked angles are α

EXAMPLE 3.2

A circle with centre O has diameter AD . The point X is on AO , and B and C are on the circle. Triangles ABX and XCO are isosceles with base angles α : $\angle BAX = \angle BXA = \alpha$ and $\angle CXO = \angle COX = \alpha$. Find α . (Australian Intermediate Mathematics Olympiad 2020)

Let the radius be 1, so $AD = 2$.

$OB = OA = OD$, so $\angle ABD = 90^\circ$ (Section 1.6) and $AB = AD \cos \alpha = 2 \cos \alpha$.

In triangle ABX , $BA = BX$, so the altitude from B bisects AX and

$$AX = 2 \cdot AB \cos \alpha = 4 \cos^2 \alpha$$

Similarly, in triangle XCO , $CX = CO = 1$, so $XO = 2 \cos \alpha$.

As $AX + XO = AO = 1$,

$$4 \cos^2 \alpha + 2 \cos \alpha - 1 = 0$$

α is a base angle, so $\cos \alpha > 0$ and $\cos \alpha = \frac{-2 + \sqrt{20}}{8} = \frac{\sqrt{5} - 1}{4} = \cos 72^\circ$.

ANSWER $\alpha = 72^\circ$

3.2 THE SINE RULE

THEOREM · SINE RULE

In any triangle, with R the radius of the circle through A, B, C :

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Use the sine rule to turn a ratio of sides into a ratio of sines. It is the natural rule when the angles are known, or when two triangles share a side. It also gives the circumradius: an equilateral triangle of side s has $R = \frac{s}{2 \sin 60^\circ} = \frac{s}{\sqrt{3}}$.

EXAMPLE 3.3

$ABCD$ is a parallelogram with $\angle D = 60^\circ$, $AD = 2$ and $AB = \sqrt{3} + 1$. M is the midpoint of AD , and the bisector of $\angle BCD$ meets BM at K . Find $\angle CKB$. (Iranian Geometry Olympiad 2018)

$\angle BCD = \angle A = 180^\circ - 60^\circ = 120^\circ$ (parallelogram), so $\angle KCB = 60^\circ$ (bisector).

$AD \parallel BC$, so $\angle MBC = \angle AMB$ (alternate angles).

In triangle ABM , $AM = 1$, $AB = \sqrt{3} + 1$ and $\angle A = 120^\circ$. So $\angle AMB + \angle ABM = 60^\circ$ and, by the sine rule,

$$\frac{\sin \angle AMB}{\sin \angle ABM} = \frac{AB}{AM} = \sqrt{3} + 1$$

Putting $\angle AMB = 45^\circ$, $\angle ABM = 15^\circ$,

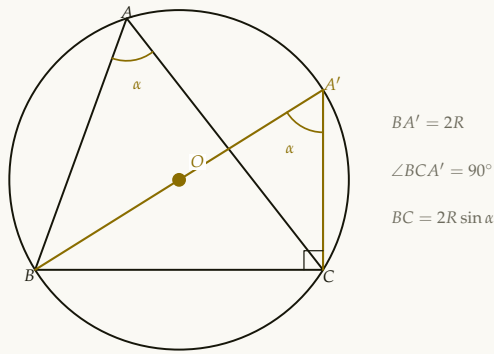
$$\frac{\sin 45^\circ}{\sin 15^\circ} = \frac{2\sqrt{2}}{\sqrt{6} - \sqrt{2}} = \frac{2}{\sqrt{3} - 1} = \sqrt{3} + 1$$

As $\angle AMB$ increases from 0° to 60° , $\angle ABM$ decreases, so the ratio increases. Hence $\angle AMB = 45^\circ$ is the only value.

So $\angle KBC = \angle MBC = 45^\circ$, and in triangle KBC ,

$$\angle CKB = 180^\circ - 60^\circ - 45^\circ$$

ANSWER 75°

**EXAMPLE 3.4**

In triangle ABC , AD is the bisector of $\angle A$, with D on BC . If $\angle B = 2\angle C$ and $CD = AB$, find $\angle A$. (RMO 2001)

Let $\angle C = \gamma$. Then $\angle B = 2\gamma$ and $\angle BAD = \angle DAC = 90^\circ - \frac{3\gamma}{2}$.

$\angle ADB$ is an exterior angle of triangle ADC , so $\angle ADB = \angle DAC + \gamma = 90^\circ - \frac{\gamma}{2}$.

By the sine rule in triangle ADC ,

$$CD = \frac{AD \sin \angle DAC}{\sin \gamma} = \frac{AD \cos \frac{3\gamma}{2}}{\sin \gamma}$$

Similarly, in triangle ADB ,

$$AB = \frac{AD \sin \angle ADB}{\sin 2\gamma} = \frac{AD \cos \frac{\gamma}{2}}{2 \sin \gamma \cos \gamma}$$

As $CD = AB$, on cancelling AD and multiplying by $2 \sin \gamma \cos \gamma$,

$$2 \cos \frac{3\gamma}{2} \cos \gamma = \cos \frac{\gamma}{2}$$

But LHS = $\cos \frac{5\gamma}{2} + \cos \frac{\gamma}{2}$, so $\cos \frac{5\gamma}{2} = 0$.

As $3\gamma < 180^\circ$, $0^\circ < \frac{5\gamma}{2} < 150^\circ$. So $\frac{5\gamma}{2} = 90^\circ$, $\gamma = 36^\circ$ and $\angle A = 180^\circ - 108^\circ$.

ANSWER 72°

3.3 THE COSINE RULE**THEOREM · COSINE RULE**

$$c^2 = a^2 + b^2 - 2ab \cos C, \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

For $C = 90^\circ$ it is Pythagoras' theorem.

Use the cosine rule when three sides are known, or two sides and the angle between them. The sign of $a^2 + b^2 - c^2$ tells whether C is acute, right or obtuse.

EXAMPLE 3.5

The sides of a convex quadrilateral are $\sqrt{a}$, $\sqrt{a+3}$, $\sqrt{a+2}$, $\sqrt{2a+5}$, in this order, and each diagonal has length $\sqrt{2a+5}$. If θ° is the difference between the largest angle and the second-largest angle, find θ . (IOQM 2026, Q27)

Let the quadrilateral be $ABCD$ with $AB = \sqrt{a}$, $BC = \sqrt{a+3}$, $CD = \sqrt{a+2}$, $DA = \sqrt{2a+5}$ and $AC = BD = \sqrt{2a+5}$.

In triangle BCD , $BC^2 + CD^2 = 2a + 5 = BD^2$, so $\angle BCD = 90^\circ$.

Let $\varphi = \angle ACD$. As $ABCD$ is convex, $\angle ACB = 90^\circ - \varphi$ and $\cos \angle ACB = \sin \varphi$.

In triangle ACD , $AC = AD$, so the altitude from A bisects CD and

$$\cos \varphi = \frac{\sqrt{a+2}}{2\sqrt{2a+5}}, \quad \sin \varphi = \frac{\sqrt{7a+18}}{2\sqrt{2a+5}}$$

By the cosine rule in triangle ABC ,

$$\begin{aligned} \cos \angle ACB &= \frac{(2a+5) + (a+3) - a}{2\sqrt{2a+5}\sqrt{a+3}} \\ &= \frac{a+4}{\sqrt{2a+5}\sqrt{a+3}} \end{aligned}$$

Putting $\cos \angle ACB = \sin \varphi$ and cancelling $\sqrt{2a+5}$,

$$2(a+4) = \sqrt{7a+18}\sqrt{a+3}$$

On squaring, $4(a+4)^2 = (7a+18)(a+3)$, i.e., $3a^2 + 7a - 10 = 0$, or $(a-1)(3a+10) = 0$. As $a > 0$, $a = 1$.

Now $AB = 1$, $BC = 2$, $CD = \sqrt{3}$, $DA = AC = BD = \sqrt{7}$.

(i) $\angle C = 90^\circ$.

(ii) $\cos B = \frac{1+4-7}{2 \cdot 1 \cdot 2} = -\frac{1}{2}$, so $\angle B = 120^\circ$.

(iii) $\angle D = \angle ACD = \varphi$ (base angles) and $0 < \cos \varphi = \frac{\sqrt{3}}{2\sqrt{7}} < \frac{1}{2}$, so $60^\circ < \angle D < 90^\circ$.

(iv) $\angle A = 150^\circ - \angle D$ (angle sum 360°), so $60^\circ < \angle A < 90^\circ$.

Hence the two largest angles are 120° and 90° .

ANSWER $\theta = 30$

3.4 AREAS WITH SINES

THEOREM · AREA OF A TRIANGLE

$$\Delta = \frac{1}{2}ab \sin C = \frac{abc}{4R} = rs,$$

where R is the circumradius, r the inradius and s half the perimeter.

The second form comes from $\sin C = \frac{c}{2R}$. The third is proved in Chapter 4. With Heron's formula (Section 2.5) they give R and r from the three sides. For the sides 13, 14, 15, $\Delta = 84$ and $s = 21$, so

$$R = \frac{13 \cdot 14 \cdot 15}{4 \cdot 84} = \frac{65}{8}, \quad r = \frac{84}{21} = 4.$$

EXAMPLE 3.6

In triangle ABC , $AB = AC$ and $\angle BAC$ is obtuse. E and F lie on AB and AC with $AE = 15$ and $AF = 10$. The area of triangle AEF is 60, and the area of $BEFC$ is 102. Find BC . (Purple Comet High School 2020)

In triangle AEF ,

$$\frac{1}{2} \cdot 15 \cdot 10 \cdot \sin A = 60, \quad \sin A = \frac{4}{5}$$

A is obtuse, so $\cos A = -\frac{3}{5}$.

Let $AB = AC = b$. The area of triangle ABC is $60 + 102 = 162$, so

$$\frac{1}{2}b^2 \cdot \frac{4}{5} = 162, \quad b^2 = 405$$

By the cosine rule,

$$BC^2 = 2b^2 - 2b^2 \cos A = 2 \cdot 405 \cdot \frac{8}{5} = 1296$$

ANSWER $BC = 36$

WHERE MARKS GO · TWO ANGLES, ONE SINE

$\sin A = \frac{4}{5}$ holds for an acute A and for an obtuse A , with $\cos A = \pm \frac{3}{5}$. Read the question again to choose. Here the acute choice gives $BC^2 = 324$, a wrong answer that looks right.

3.5 IDENTITIES IN A TRIANGLE

In a triangle, $A + B + C = 180^\circ$, so $\sin(A + B) = \sin C$ and $\cos(A + B) = -\cos C$. If two equations give $\sin x$

and $\cos x$, squaring and adding them removes x , since $\sin^2 x + \cos^2 x = 1$.

Two identities that follow from $A + B + C = 180^\circ$:

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

when no angle is 90° , and

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

EXAMPLE 3.7

In triangle ABC , $\cos A = 2 \cos^3 B$ and $\sin A = 2 \sin^3 B$. Find $\angle C$. (Czech and Slovak National Round 1956)

On squaring and adding,

$$1 = 4(\cos^6 B + \sin^6 B)$$

Putting $x = \cos^2 B$, $y = \sin^2 B$, $x + y = 1$ in $x^3 + y^3 = (x + y)^3 - 3xy(x + y)$,

$$\cos^6 B + \sin^6 B = 1 - 3u, \quad u = \sin^2 B \cos^2 B$$

So $1 = 4 - 12u$, $u = \frac{1}{4}$ and $\sin^2 2B = 4u = 1$.

As $0^\circ < 2B < 360^\circ$, $2B = 90^\circ$ or 270° , i.e., $B = 45^\circ$ or $B = 135^\circ$.

(i) $B = 135^\circ$: $\cos A = 2 \cos^3 B < 0$ gives $A = 135^\circ$, so $A + B > 180^\circ$, not possible.

(ii) $B = 45^\circ$: $\cos A = \sin A = 2\left(\frac{\sqrt{2}}{2}\right)^3 = \frac{\sqrt{2}}{2}$, so $A = 45^\circ$.

ANSWER $C = 90^\circ$

EXAMPLE 3.8

If $\sin \theta + \cos \theta + \tan \theta + \cot \theta + \sec \theta + \operatorname{cosec} \theta = 7$, then $\sin 2\theta = a - b\sqrt{7}$, where a and b are positive integers. Find $a + b$. (New York State Maths League 1988)

Let $s = \sin \theta + \cos \theta$ and $p = \sin \theta \cos \theta$. Then $s^2 = 1 + 2p$ and

$$\tan \theta + \cot \theta = \frac{\sin^2 \theta + \cos^2 \theta}{p} = \frac{1}{p}$$

$$\sec \theta + \operatorname{cosec} \theta = \frac{\cos \theta + \sin \theta}{p} = \frac{s}{p}$$

So the given equation becomes

$$s + \frac{1+s}{p} = 7, \quad \text{with } p = \frac{(s-1)(s+1)}{2}$$

Now $\frac{1+s}{p} = \frac{2}{s-1}$, so

$$s + \frac{2}{s-1} = 7, \quad s^2 - 8s + 9 = 0, \quad s = 4 \pm \sqrt{7}$$

But $s = \sin \theta + \cos \theta \leq \sqrt{2} < 4 + \sqrt{7}$, so $s = 4 - \sqrt{7}$ and

$$\sin 2\theta = s^2 - 1 = 22 - 8\sqrt{7}$$

ANSWER $a + b = 30$

EXERCISES 3

3.1 Find $\tan 15^\circ + \cot 15^\circ$.

3.2 Find $\cos 36^\circ - \cos 72^\circ$.

3.3 Find $\sin 18^\circ \cos 36^\circ$.

3.4 In triangle ABC , $a = 5$, $b = 8$ and $\angle C = 60^\circ$. Find c .

3.5 Find the largest angle of the triangle with sides 3, 5, 7.

3.6 In a triangle, the side opposite an angle of 30° is 6. Find the radius of the circle through its vertices.

3.7 In triangle ABC , $\angle A = 45^\circ$, $\angle B = 60^\circ$ and $BC = 6$. Find AC^2 .

3.8 Triangle ABC has $BC = 4$, $CA = 5$ and $AB = 6$. Find $\angle C : \angle A$.

3.9 Two sides of a triangle are 8 and 10, and the angle between them is 150° . Find the area.

3.10 If $\sin x + \cos x = \frac{7}{5}$ with $0^\circ < x < 90^\circ$, find $\tan x + \cot x$.

CHAPTER 04 CIRCLES

A circle in a figure gives equal angles and equal products of lengths. This chapter covers angles in a circle, cyclic quadrilaterals and Ptolemy's theorem, tangents, the power of a point, the incircle, circles that touch, and the centres of a triangle. Many questions have a circle that is not drawn; finding it is often the whole solution.

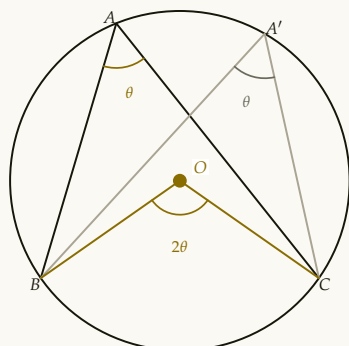
4.1 ANGLES IN A CIRCLE

THEOREM · INSCRIBED ANGLES

The angle at the centre is twice the angle at the circumference on the same arc.

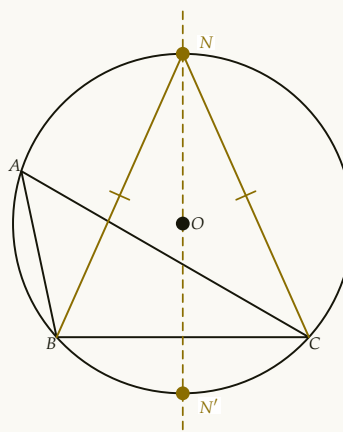
Angles on the same arc, on the same side of the chord, are equal.

The angle in a semicircle is 90° .



angle at the centre
is twice the angle
at the circumference

The perpendicular bisector of a chord BC passes through the centre, so it meets the circle at the midpoints of the two arcs BC . Each arc midpoint is at the same distance from B and from C .



N : midpoint of arc BAC

$NB = NC$

N' : midpoint of arc BC

EXAMPLE 4.1

In triangle ABC , $\angle CAB = 80^\circ$. The perpendicular bisector of BC meets the circumcircle of ABC at N , where A and N lie on the same side of the chord BC . Find $\angle NBC$. (IOQM 2026, Q5)

N lies on the perpendicular bisector of BC , so $NB = NC$.

N and A lie on the same arc BC , so $\angle BNC = \angle BAC = 80^\circ$.

So, in isosceles $\triangle NBC$,

$$\angle NBC = \frac{180^\circ - 80^\circ}{2}$$

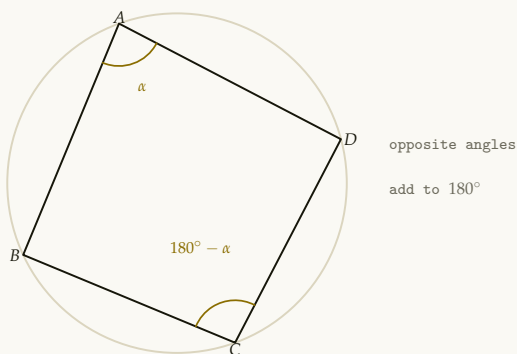
ANSWER 50°

4.2 CYCLIC QUADRILATERALS

THEOREM · CYCLIC QUADRILATERALS

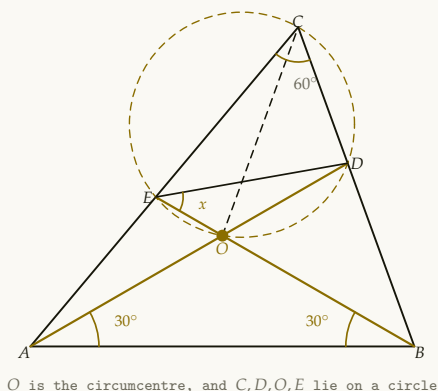
If $ABCD$ is cyclic, its opposite angles add to 180° , and $\angle ACB = \angle ADB$.

Conversely, $ABCD$ is cyclic if two opposite angles add to 180° , or if a side AB makes equal angles $\angle ACB = \angle ADB$ at C and D on the same side of AB .



KEY IDEA · FIND THE HIDDEN CIRCLE

When an angle chase stops, look for four points on a circle, or for the circumcentre. Once the circle is found, each inscribed angle is equal to another one.



EXAMPLE 4.2

In triangle ABC , $\angle CAB = 50^\circ$ and $\angle ABC = 70^\circ$. D and E are on the sides BC and AC with $\angle ABE = \angle DAB = 30^\circ$. If $\angle DEB = x^\circ$, find x . (IOQM 2026, Q30)

$$\angle C = 180^\circ - 50^\circ - 70^\circ = 60^\circ.$$

Let O be the circumcentre of ABC . Then $\angle AOB = 2\angle C = 120^\circ$ (angle at the centre is twice the angle at the circumference).

$OA = OB$, so $\angle OAB = \angle OBA = 30^\circ$. Hence O is the point of intersection of AD and BE .

$\angle DOE = \angle AOB = 120^\circ$ (vertically opposite) and $\angle DCE = 60^\circ$. Their sum is 180° , so C, D, O, E are concyclic.

In this circle, $\angle DEO$ and $\angle DCO$ stand on the same chord DO , so $\angle DEB = \angle DCO$.

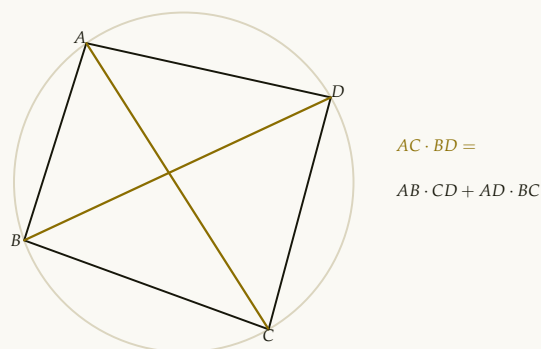
$OB = OC$ and $\angle OBC = 70^\circ - 30^\circ = 40^\circ$, so $\angle OCB = 40^\circ$.

ANSWER $x = 40$

THEOREM · PTOLEMY

In a cyclic quadrilateral $ABCD$,

$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$



For example, in a regular pentagon with side s and diagonal d , Ptolemy for four of its vertices gives $d^2 = s^2 + sd$. So $\frac{d}{s} = \frac{1+\sqrt{5}}{2}$.

4.3 TANGENTS

THEOREM · TANGENT FACTS

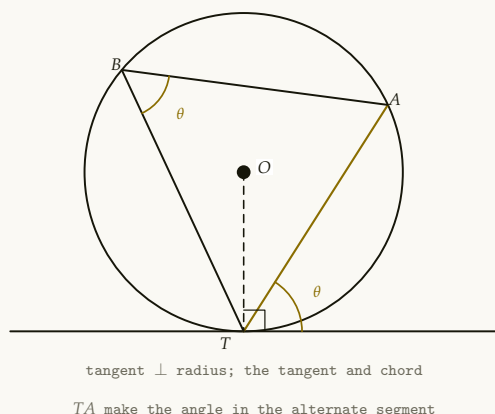
A tangent is perpendicular to the radius at the point of contact.

The two tangents from an outside point are equal.

The angle between a tangent and a chord equals the inscribed angle on the other side of the chord.

A convex quadrilateral $ABCD$ has an incircle exactly when $AB + CD = BC + DA$.

Let the tangents from P touch the circle, centre O , at T_1 and T_2 . The quadrilateral OT_1PT_2 has two right angles, so $\angle T_1PT_2 + \angle T_1OT_2 = 180^\circ$. Also OP bisects both angles and is the perpendicular bisector of T_1T_2 .



EXAMPLE 4.3

A circle passes through the vertex C of a rectangle $ABCD$ and touches the sides AB and AD at M and N . If the distance from C to the line MN is 5, find the area of $ABCD$. (INMO 1994)

Let A be the origin, AB along the x -axis and AD along the y -axis, with $C = (p, q)$. Area = pq .

The circle touches both axes, so its centre is (r, r) and radius r . Then $M = (r, 0)$ and $N = (0, r)$.

C lies on the circle, so

$$(p - r)^2 + (q - r)^2 = r^2$$

MN is $x + y = r$, so the distance of C from MN is $\frac{|p+q-r|}{\sqrt{2}}$. On expanding and using the circle equation,

$$\begin{aligned}(p + q - r)^2 &= (p - r)^2 + (q - r)^2 - r^2 + 2pq \\ &= 2pq\end{aligned}$$

$$\text{So } 5^2 = \frac{(p+q-r)^2}{2} = \frac{2pq}{2} = pq.$$

ANSWER 25

EXAMPLE 4.4

Each side of a rhombus is 5 cm and one of its angles is 60° . A circle lies entirely inside the rhombus, and its area is $n \text{ cm}^2$, where n is a positive integer. Find the number of possible values of n . (Math Prize for Girls 2010)

The largest circle inside the rhombus touches all four sides. Its diameter is the distance between two opposite sides, i.e., the height of the rhombus

$$5 \sin 60^\circ = \frac{5\sqrt{3}}{2}$$

So the largest radius is $\frac{5\sqrt{3}}{4}$ and the largest area is $\frac{75\pi}{16} \approx 14.7$.

Every smaller circle also fits, so the area can be any positive value up to 14.7. Hence $n = 1, 2, \dots, 14$.

ANSWER 14

4.4 THE POWER OF A POINT

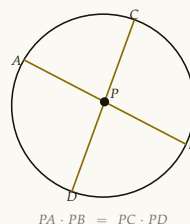
When a line through a point P meets a circle at two points, the product of the two distances from P is the same for every such line; it is called the **power** of P .

THEOREM · POWER OF A POINT

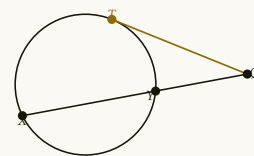
If two lines through P meet a circle at A, B and at C, D , then

$$PA \cdot PB = PC \cdot PD,$$

whether P is inside or outside the circle. If a line from P touches the circle at T , then $PT^2 = PA \cdot PB$. Both products equal $|d^2 - r^2|$, where d is the distance from P to the centre.



$$PA \cdot PB = PC \cdot PD$$



$$QT^2 = QX \cdot QY$$

The products come from similar triangles: for chords, PAC and PDB have equal angles at A and D (same arc), and vertically opposite angles at P .

EXAMPLE 4.5

$ABCD$ is a cyclic quadrilateral in a circle of radius 4, and $AC \perp BD$ at E . Find $EA^2 + EB^2 + EC^2 + ED^2$. (RMO 1992)

$\angle AEB = \angle CED = 90^\circ$, so

$$EA^2 + EB^2 = AB^2, \quad EC^2 + ED^2 = CD^2$$

In right $\triangle AED$, $\angle ADB + \angle CAD = 90^\circ$.

A chord is $2R$ times the sine of an inscribed angle on it (sine rule), so

$$AB = 2R \sin \angle ADB$$

$$CD = 2R \sin \angle CAD = 2R \cos \angle ADB$$

$$\text{Hence } AB^2 + CD^2 = 4R^2 = 4 \cdot 16.$$

ANSWER 64

EXAMPLE 4.6

The medians AM and BE of triangle ABC meet at G . The points G, M, E, C lie on one circle. If $BE = AM = 3$, find AB^2 . (Yasinsky Geometry Olympiad 2017)

G is the centroid, so $AG = \frac{2}{3}AM = 2$ and $BG = 1$.

Line AC meets the circle through G, M, E, C at E, C , and line AM meets it at G, M . So, by the power of A ,

$$AE \cdot AC = AG \cdot AM$$

$$\frac{1}{2}AC^2 = 2 \cdot 3, \quad AC^2 = 12$$

Similarly, by the power of B , $BM \cdot BC = BG \cdot BE$, so $BC^2 = 12$.

By the median formula (Section 2.4) for AM ,

$$AB^2 + AC^2 = 2AM^2 + \frac{1}{2}BC^2$$

$$AB^2 + 12 = 18 + 6$$

ANSWER $AB^2 = 12$

4.5 THE INCIRCLE

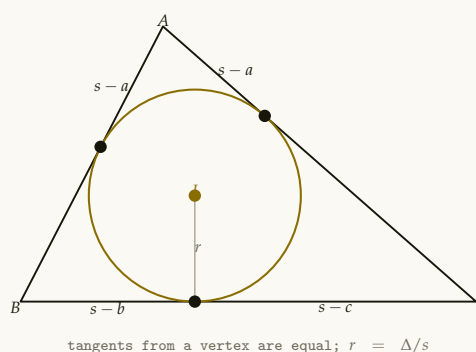
THEOREM · INCIRCLE FACTS

The incircle, with centre I and radius r , touches BC, CA, AB at D, E, F . With s half the perimeter,

$$AE = AF = s - a, \quad BD = BF = s - b,$$

$$CD = CE = s - c.$$

Joining I to the vertices cuts the triangle into three triangles of height r , so $\Delta = rs$. In a right triangle with hypotenuse c , $r = s - c = \frac{a+b-c}{2}$.



EXAMPLE 4.7

Triangle ABC has $AB = 3$, $BC = 4$ and $AC = 5$, and I is its incentre. Find $AI \cdot BI \cdot CI$. (Math Prize for Girls 2011)

$\angle B = 90^\circ$ (as $3^2 + 4^2 = 5^2$) and $s = 6$, so $r = s - AC = 1$.

vertex	tangent length	$IX^2 = r^2 + t^2$
A	$s - BC = 2$	$1 + 4 = 5$
B	$s - AC = 1$	$1 + 1 = 2$
C	$s - AB = 3$	$1 + 9 = 10$

Each IX is the hypotenuse of a right triangle with legs r and tangent length t . Hence

$$AI \cdot BI \cdot CI = \sqrt{5 \cdot 2 \cdot 10} = \sqrt{100}$$

ANSWER 10

EXAMPLE 4.8

$ABCD$ is a quadrilateral with $AB \parallel CD$, $AB \perp AD$ and $AB = 3CD$, and its area is 1728. A circle touches all four sides. Find its radius. (RMO 2006)

Let the radius be r , $CD = x$ and $AB = 3x$.

AD is perpendicular to the parallel sides AB , CD , and the circle touches both, so $AD = 2r$.

$ABCD$ has an incircle, so $AB + CD = AD + BC$, i.e., $BC = 4x - 2r$.

Draw $CK \perp AB$. In right $\triangle CKB$, $KB = AB - CD = 2x$, $CK = 2r$ and hypotenuse $BC = 4x - 2r$, so

$$(4x - 2r)^2 = (2x)^2 + (2r)^2$$

On simplifying, $12x^2 = 16xr$, so $x = \frac{4}{3}r$ (as $x \neq 0$).

Area = $\frac{1}{2}(AB + CD) \cdot AD = 4x \cdot r = \frac{16}{3}r^2 = 1728$, so $r^2 = 324$.

ANSWER $r = 18$

EXAMPLE 4.9

The incircle of triangle ABC has radius 3 and touches BC at D . The altitude from A to BC is 15, and $BD^2 + CD^2 = 33$. Find BC^2 . (CMIMC Team 2020)

Let $a = BC$. Then $\Delta = rs = 3s$ and also $\Delta = \frac{1}{2} \cdot a \cdot 15$.

So $s = \frac{5}{2}a$ and $s - a = \frac{3}{2}a$.

Also $BD = s - b$, $CD = s - c$. By Heron's formula, with $\Delta = 3s$,

$$9s^2 = s(s - a) \cdot BD \cdot CD$$

$$BD \cdot CD = \frac{9s}{s - a} = \frac{9 \cdot \frac{5}{2}a}{\frac{3}{2}a} = 15$$

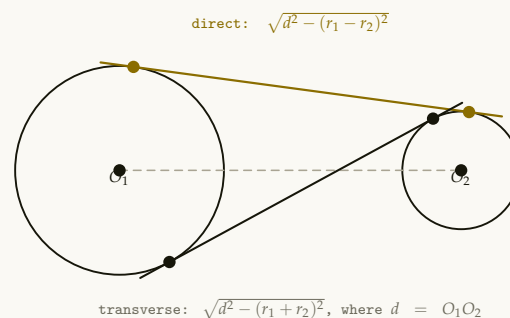
As $BD + CD = a$,

$$a^2 = BD^2 + CD^2 + 2BD \cdot CD = 33 + 30$$

ANSWER 63

4.6 CIRCLES THAT TOUCH

Two circles with radii r_1, r_2 and centres d apart touch from outside when $d = r_1 + r_2$, and from inside when $d = |r_1 - r_2|$. The point of contact lies on the line of centres.



An enlargement from a point maps a circle to a circle. Circles inside an angle that touch both arms are enlargements of each other from the vertex.

EXAMPLE 4.10

Five marbles of different sizes are in a conical funnel. Each marble touches the next one and the wall of the funnel. The smallest has radius 8 mm and the largest 18 mm. Find the radius of the middle marble. (Crux Mathematicorum, Mathematic MA-129)

Let the enlargement from the apex of the cone take marble 1 to marble 2. It maps the funnel to itself. So the image of marble 2 touches the funnel and marble 2 on the far side, i.e., it is marble 3; and so on.

So each radius is k times the one before it, i.e., the radii are $r_1, r_1k, \dots, r_1k^4$. With $r_1 = 8$ and $r_1k^4 = 18$, the middle one is

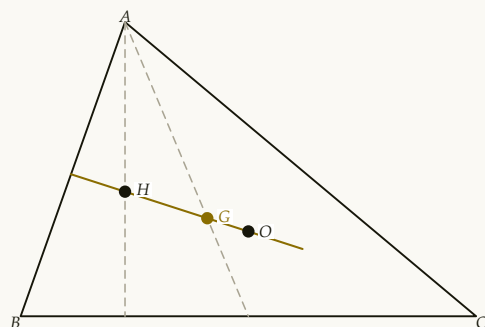
$$r_1k^2 = \sqrt{r_1 \cdot r_1k^4} = \sqrt{8 \cdot 18} = \sqrt{144}$$

ANSWER 12 mm

4.7 CENTRES OF A TRIANGLE

THEOREM · FOUR CENTRES

The medians meet at the centroid G , which divides each median $2 : 1$. The perpendicular bisectors meet at the circumcentre O , the angle bisectors at the incentre I , the altitudes at the orthocentre H . O, G, H lie on one line with $OG : GH = 1 : 2$ (the Euler line), and $AH = 2R |\cos A|$.



circumcentre, centroid, orthocentre lie on
the Euler line, with $OG : GH = 1 : 2$

In an equilateral triangle the four centres are the same point. In a right triangle, O is the midpoint of the hypotenuse and H is the vertex of the right angle.

EXERCISES 4

- 4.1 A chord AB makes an angle of 130° at the centre of a circle. C is on the major arc. Find $\angle ACB$.
- 4.2 $ABCD$ is a cyclic quadrilateral with $\angle A = 3\angle C$. Find $\angle A$.
- 4.3 Chords AB and CD of a circle meet at P , with $AP = 4$, $PB = 6$ and $CP = 3$. Find CD .
- 4.4 A point is 13 from the centre of a circle of radius 5. Find the length of a tangent from it.
- 4.5 Find the inradius and the circumradius of the triangle with sides 5, 12, 13.

- 4.6 A quadrilateral $ABCD$ has an incircle, with $AB = 7$, $BC = 8$ and $CD = 10$. Find DA .
- 4.7 From an outside point P , a line meets a circle at A and B , with $PA = 4$ and $AB = 5$. Find the length of a tangent from P .
- 4.8 P is on the arc AB , not containing C , of the circumcircle of an equilateral triangle ABC . If $PA = 3$ and $PB = 5$, find PC .
- 4.9 In triangle ABC , $\angle A = 60^\circ$ and the circumradius is 10. Find AH , where H is the orthocentre.
- 4.10 $ABCD$ and $DEFG$ are rectangles, with E on the side AD , G on the side CD , and F the incentre of triangle ABC . Find the ratio of the area of $ABCD$ to the area of $DEFG$. (*Crux Mathematicorum, Mathematic MA-116*)
- 4.11 Three semicircles of radius 1 lie inside an equilateral triangle. The diameter of each lies along a different side, each touches the other two, and each also touches a second side of the triangle. Find the side of the triangle. (*UK Intermediate Maths Challenge 2019*)
- 4.12 One way to pack a 100×100 square with 10000 circles of diameter 1 is in 100 rows of 100. If instead the circles are packed so that the centres of any three touching circles form an equilateral triangle, what is the largest number of extra circles that fit? (*Cayley Contest 1998*)

CHAPTER 05 COORDINATES AND PROBLEM SOLVING

Some geometry questions are fastest with an idea, and some with plain calculation. This chapter covers coordinates, lattice points and Pick's theorem, and shortest paths by reflection. It then lists the geometry questions of IOQM 2026, gives a chart for choosing the method, notes common mistakes, and ends with a mixed set.

5.1 COORDINATES

RULES · COORDINATE FACTS

Distance: $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

The point dividing PQ in the ratio $m : n$ is

$$\frac{nP + mQ}{m + n}.$$

Two lines with slopes m_1, m_2 are perpendicular when $m_1 m_2 = -1$.

Area of a triangle (shoelace):

$$\frac{1}{2} |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|.$$

Choose the axes to make the numbers small: put a right angle at the origin with its sides on the axes, or

put a line of symmetry on an axis.

EXAMPLE 5.1

Solve Example 2.6 (IOQM 2026, Q24) again, with coordinates: $ABCD$ is a rectangle, E is on BD with $AE \perp BD$, $AE = 12$ and $CE = \sqrt{193}$. Find the area of $ABCD$.

Let $A = (0,0)$, $B = (p,0)$, $C = (p,q)$, $D = (0,q)$ and $d = BD = \sqrt{p^2 + q^2}$.

$2[ABD] = pq = d \cdot AE$, so $AE = \frac{pq}{d}$.

The foot of the perpendicular from A to BD is

$$E = \left(\frac{pq^2}{d^2}, \frac{p^2q}{d^2} \right)$$

(on BD : $\frac{x}{p} + \frac{y}{q} = 1$; slope of $AE \times$ slope of $BD = \frac{p}{q} \cdot \left(-\frac{q}{p}\right) = -1$). Then $p - \frac{pq^2}{d^2} = \frac{p^3}{d^2}$ and $q - \frac{p^2q}{d^2} = \frac{q^3}{d^2}$, so

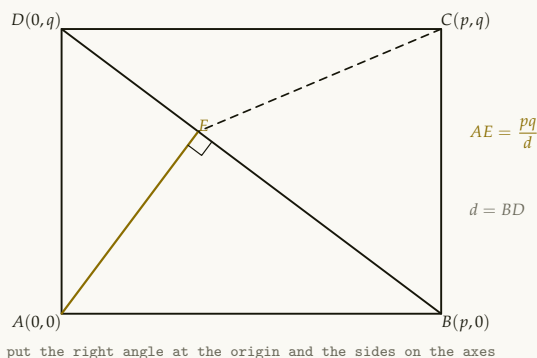
$$CE^2 = \frac{p^6 + q^6}{d^4}$$

Now $p^6 + q^6 = (p^2 + q^2)^3 - 3p^2q^2(p^2 + q^2) = d^6 - 3p^2q^2d^2$, so

$$CE^2 = d^2 - 3\left(\frac{pq}{d}\right)^2 = d^2 - 3AE^2$$

Putting $AE = 12$, $CE = \sqrt{193}$, we get $193 = d^2 - 432$, so $d = 25$. The required area is $pq = AE \cdot d = 12 \cdot 25$.

ANSWER 300



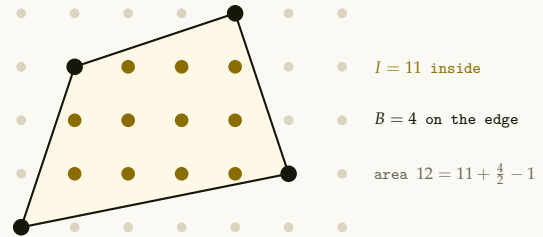
The coordinates take longer than the median idea of Section 2.4, but they need no idea at all. When no idea comes in a few minutes, calculate.

5.2 LATTICE POINTS AND PICK'S THEOREM

THEOREM · LATTICE POINTS

The segment from $(0,0)$ to (a,b) has $\gcd(a,b) + 1$ lattice points, counting both ends.

Pick: a polygon with lattice vertices, I lattice points inside and B on its boundary, has area $I + \frac{B}{2} - 1$.



For example, the triangle with vertices $(0,0)$, $(6,0)$, $(0,4)$ has area 12 and $B = 6 + 4 + \gcd(6,4) = 12$ boundary points, so $I = 12 - 6 + 1 = 7$ points inside.

EXAMPLE 5.2

The 16 small squares of a 4×4 grid each have side 1. How many pairs of vertices (points where the grid lines cross) are an integer distance apart? (UK Senior Maths Challenge 2013)

Let the 25 vertices be (x,y) , $x, y \in \{0,1,2,3,4\}$. Two vertices with differences d_x, d_y are $\sqrt{d_x^2 + d_y^2}$ apart.

Same row or column. Every such pair works ($d_x = 0$ or $d_y = 0$). A row or column has $\binom{5}{2} = 10$ pairs:

$$5 \cdot 10 + 5 \cdot 10 = 100$$

Neither. Now $1 \leq d_x, d_y \leq 4$, and the only square $d_x^2 + d_y^2$ is $3^2 + 4^2 = 5^2$. A difference k fits in $5 - k$ places:

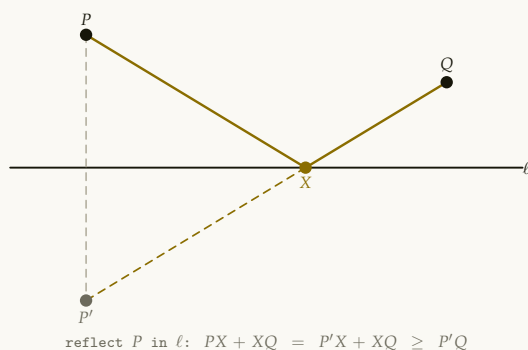
differences	places	pairs
$d_x = 3, d_y = 4$	$2 \cdot 1$	2 each way
$d_x = 4, d_y = 3$	$1 \cdot 2$	2 each way

Each segment can slope up or down, so these give $2 \cdot (2 + 2) = 8$ pairs.

ANSWER $100 + 8 = 108$

5.3 SHORTEST PATHS BY REFLECTION

To make $PX + XQ$ least with X on a line ℓ and P, Q on the same side of ℓ , reflect P in ℓ to P' . Then $PX = P'X$, and $P'X + XQ$ is least when X is on the segment $P'Q$.



EXAMPLE 5.3

Find the least value of $\sqrt{x^2 + 9} + \sqrt{(8-x)^2 + 9}$ for real x .

Let $A = (0, 3)$, $B = (8, 3)$, and $P = (x, 0)$ on the x -axis. Then $PA = \sqrt{x^2 + 9}$ and $PB = \sqrt{(8-x)^2 + 9}$.

Reflect B in the x -axis to $B' = (8, -3)$, so $PB = PB'$:

$$PA + PB' \geq AB' = \sqrt{8^2 + 6^2} = 10$$

(triangle inequality). Equality holds when P is on AB' , i.e., $P = (4, 0)$, the midpoint of AB' .

ANSWER 10

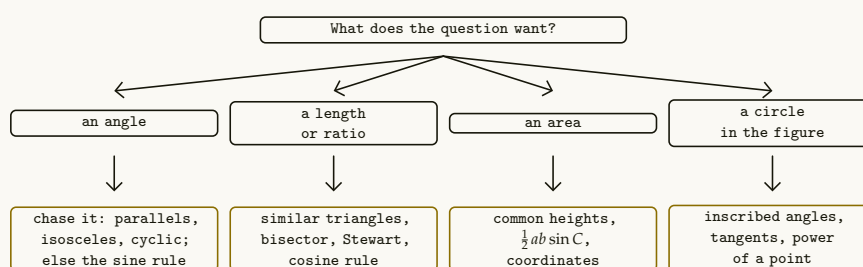
5.4 IOQM 2026 GEOMETRY QUESTIONS

Seven of the thirty questions of IOQM 2026 were geometry. Each is solved in this part; the last column gives the example.

Q	QUESTION AND IDEA	EXAMPLE
5	arc midpoint: inscribed angles	4.1
8	trapezium: parallelogram, midpoint	1.7
13	right isosceles: $\tan 75^\circ$	3.1
19	two hypotenuses: the median	1.6
24	rectangle: Apollonius, coordinates	2.6, 5.1
27	equal diagonals: cosine rule	3.5
30	two 30° lines: hidden circle	4.2

5.5 CHOOSING THE METHOD

A question in the paper does not say which method it wants. Read it once for what is asked, and use the chart and the table below. If no method comes in a few minutes, draw an accurate figure, guess the answer from it, and then calculate.



THE QUESTION ASKS	METHOD TO TRY FIRST	SECTION
an angle	parallels, isosceles, a hidden circle; else the sine rule	1.1, 1.2, 4.2, 3.2
a length in a right triangle	Pythagoras, special triangles, the altitude	1.5, 2.2
a length from ratios	similar triangles, bisector, Stewart, Apollonius	2.1, 2.3, 2.4
three sides are given	cosine rule, Heron, R and r	2.5, 3.3, 3.4
a ratio of areas	common heights, a shared angle, Menelaus	2.5, 2.6
a circle in the figure	inscribed angles, tangents, power of a point	4.1 to 4.5
a rectangle or many right angles	coordinates	5.1
lattice points or a shortest path	gcd and Pick; reflection	5.2, 5.3

5.6 COMMON MISTAKES

Students often lose marks on these points.

- (i) Using two sides and an angle not between them as a congruence test.
- (ii) Taking the acute angle when $\sin A$ is given and A is obtuse.
- (iii) Giving vertex A the tangent length $s - b$; from A it is $s - a$.
- (iv) In Stewart's theorem, pairing a side with the near part of the base instead of the far part.
- (v) Trusting the figure: points that look collinear, or angles that look equal.
- (vi) Treating a ratio of areas as the ratio of lengths; areas go with the square.
- (vii) Forgetting that two circles can also touch from inside.
- (viii) Using decimals such as $\sqrt{3} \approx 1.73$, which can give a wrong integer.
- (ix) Not checking the triangle inequality for lengths found from a condition.

5.7 A MIXED PROBLEM SET

As in the actual paper, the exercises below are not arranged by chapter. Before solving, decide from the table in Section 5.5 which method the question needs. Answers with one-line hints are at the end of this part.

EXERCISES 5

- 5.1 A regular polygon has 20 diagonals. How many sides does it have?
- 5.2 How many lattice points lie on the segment from $(0,0)$ to $(12,18)$, counting both ends?
- 5.3 How many lattice points lie strictly inside the triangle with vertices $(0,0)$, $(7,0)$ and $(0,5)$?
- 5.4 Find the area of the triangle with vertices $(1,2)$, $(4,6)$ and $(7,3)$.
- 5.5 Find the least value of $\sqrt{x^2 + 1} + \sqrt{(x-4)^2 + 4}$ for real x .
- 5.6 In triangle ABC , $AB = AC = 1$ and $\angle A = 36^\circ$. Find BC .
- 5.7 Find the distance between the incentre and the circumcentre of the triangle with sides 6, 8, 10.
- 5.8 $ABCD$ is a square of side 6 and E is the midpoint of BC . AE meets BD at F . Find the area of triangle BEF .
- 5.9 From a point P , the tangent PT to a circle has length 12. A line through P meets the circle at A and B , with $PA = 8$. Find AB .
- 5.10 $ABCD$ is a cyclic quadrilateral with $AB = BC$ and $\angle ABC = 100^\circ$. Find $\angle ADB$.
- 5.11 Find the inradius of the triangle with sides 9, 10, 17.
- 5.12 In triangle ABC of area 48, D is on BC with $BD : DC = 1 : 2$, and E is on AD with $AE : ED = 3 : 1$. Find the area of triangle BED .
- 5.13 In triangle ABC , $\angle A = 40^\circ$, and the bisectors of $\angle B$ and $\angle C$ meet at I . Find $\angle BIC$.

PART III ANSWERS TO THE EXERCISES

Every answer below was checked by computer, by exact calculation or with coordinates. The hint gives only the first idea of the solution; if it is enough, complete the question yourself before reading further.

CHAPTER 1 · ANGLES, TRIANGLES AND CONGRUENCE

- 1.1 30 each exterior angle is 12° : $\frac{360}{12}$.
- 1.2 90 $(n-2) \cdot 180 = 2340$ gives $n = 15$; $\frac{15 \cdot 12}{2}$.
- 1.3 23 $a_1 + a_2 + a_3 = a_3 + a_4 + a_5 = a_5 + a_6 + a_1$ gives $a_5 = 4, a_6 = 5$.
- 1.4 36° with $\angle A = x$: $\angle BDC = 2x = \angle C = \angle B$, so $5x = 180$.
- 1.5 150° $AD = AE$ and $\angle DAE = 30^\circ$ give $\angle AED = 75^\circ$; the same at B : $360 - 60 - 75 - 75$.
- 1.6 7 longest side at most 7: $(1, 7, 7), (2, 6, 7), (3, 5, 7), (4, 4, 7), (3, 6, 6), (4, 5, 6), (5, 5, 5)$.
- 1.7 4 $x = 22, 23, 24, 25$: longest side 24 needs $x^2 > 476$, longest side x needs $x^2 < 676$.
- 1.8 4 British flag: $1 + 64 = 49 + PD^2$.
- 1.9 65° $MC = MB$, so $\angle B = 25^\circ$.
- 1.10 24 Varignon: the perimeter is the sum of the diagonals.
- 1.11 1 : 2 draw $MQ \parallel BP$ to AC : the midpoint theorem twice gives $AP = PQ = QC$.
- 1.12 $25(2 - \sqrt{3})$ with s the side of $ABCD$, $EFGH$ is a square with diagonal $s(1 + \sqrt{3})$, so its area is $s^2(2 + \sqrt{3})$.

CHAPTER 2 · SIMILARITY, LENGTHS AND AREAS

- 2.1 2 similar triangles give the side $\frac{ab}{a+b} = \frac{18}{9}$.
- 2.2 63 $[ADE] = \left(\frac{2}{5}\right)^2 \cdot 75 = 12$.
- 2.3 39 $h^2 = 4 \cdot 9$, so $h = 6$; $\frac{1}{2} \cdot 13 \cdot 6$.
- 2.4 5 $CF : FA = CE : AB = 1 : 2$ and $[ACE] = 15$.
- 2.5 30 $BD = 4, DC = 6; AD^2 = bc - mn = 54 - 24$.
- 2.6 7 $m^2 = \frac{2 \cdot 49 + 2 \cdot 81 - 64}{4} = 49$.
- 2.7 12 Heron: area 84, so $h = \frac{2 \cdot 84}{14}$.
- 2.8 76 Heron: $(a+b+c)(b+c-a)(c+a-b)(a+b-c) = 16 \cdot 114^2$; only the sides 19, 20, 37 work.
- 2.9 3 : 1 Ceva: $\frac{2}{3} \cdot \frac{1}{2} \cdot \frac{AF}{FB} = 1$.
- 2.10 65 $B = (0, 0), C = (112, 0), A = (40, 96)$; then

$$N = (0, 33) \text{ and } ND^2 = 56^2 + 33^2.$$

- 2.11 4 : 3 FC is midway between AB and ED , so Q is the midpoint of PE ; with side 2 the areas are $\sqrt{3}$ and $\frac{3\sqrt{3}}{4}$.

CHAPTER 3 · TRIGONOMETRY IN GEOMETRY

- 3.1 4 $\frac{1}{\sin 15^\circ \cos 15^\circ} = \frac{2}{\sin 30^\circ}$.
- 3.2 $\frac{1}{2} \frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4}$.
- 3.3 $\frac{1}{4} \frac{\sqrt{5}-1}{4} \cdot \frac{\sqrt{5}+1}{4} = \frac{4}{16}$.
- 3.4 7 $c^2 = 25 + 64 - 40$.
- 3.5 120° $\cos C = \frac{9+25-49}{30} = -\frac{1}{2}$.
- 3.6 6 $2R = \frac{6}{\sin 30^\circ} = 12$.
- 3.7 54 $AC = \frac{6 \sin 60^\circ}{\sin 45^\circ} = 3\sqrt{6}$.
- 3.8 2 : 1 $\cos A = \frac{3}{4}$ and $\cos C = \frac{1}{8} = 2 \cos^2 A - 1$.
- 3.9 20 $\frac{1}{2} \cdot 8 \cdot 10 \cdot \sin 150^\circ$.
- 3.10 $\frac{25}{12} \sin x \cos x = \frac{1}{2} \left(\frac{49}{25} - 1 \right) = \frac{12}{25}$.

CHAPTER 4 · CIRCLES

- 4.1 65° half the angle at the centre.
- 4.2 135° $\angle A + \angle C = 180^\circ$ and $\angle A = 3\angle C$.
- 4.3 11 $PD = \frac{4 \cdot 6}{3} = 8$.
- 4.4 12 $\sqrt{13^2 - 5^2}$.
- 4.5 2 and $\frac{13}{2}$ $r = \frac{5+12-13}{2}$; R is half the hypotenuse.
- 4.6 9 $AB + CD = BC + DA$.
- 4.7 6 $PT^2 = PA \cdot PB = 4 \cdot 9$.
- 4.8 8 Ptolemy in $APBC$: $PC \cdot AB = PA \cdot BC + PB \cdot CA$, so $PC = PA + PB$.
- 4.9 10 $AH = 2R \cos 60^\circ$.
- 4.10 2 with legs w, h , hypotenuse d and $r = \frac{w+h-d}{2}$: $(w-r)(h-r) = \frac{wh}{2}$.
- 4.11 $2\sqrt{3}$ each centre is $\frac{2}{\sqrt{3}}$ from a vertex, and the centres are 2 apart.
- 4.12 1443 115 rows fit, 58 of 100 and 57 of 99: 11443 circles.

CHAPTER 5 · COORDINATES AND PROBLEM SOLVING

- 5.1 8 $\frac{n(n-3)}{2} = 20$.
- 5.2 7 $\gcd(12, 18) + 1$.
- 5.3 12 area $\frac{35}{2}$, $B = 13$; Pick: $\frac{35}{2} - \frac{13}{2} + 1$.

$$5.4 \quad \frac{21}{2} \text{ shoelace: } \frac{1}{2} |1 \cdot 3 + 4 \cdot 1 + 7 \cdot (-4)|.$$

$$5.5 \quad 5 \text{ reflect } (4, 2) \text{ to } (4, -2); \text{ its distance from } (0, 1).$$

$$5.6 \quad \frac{\sqrt{5}-1}{2} \text{ the bisector of } \angle B \text{ cuts off a similar triangle: } BC^2 = 1 - BC.$$

$$5.7 \quad \sqrt{5} \text{ legs on the axes: } I = (2, 2) \text{ and } O = (3, 4).$$

$$5.8 \quad 3 \quad A = (0, 0), C = (6, 6): F = (4, 2).$$

$$5.9 \quad 10 \quad PB = \frac{144}{8} = 18.$$

$$5.10 \quad 40^\circ \quad \angle ADB = \angle ACB = \frac{180-100}{2}.$$

$$5.11 \quad 2 \quad s = 18, \text{ area } \sqrt{18 \cdot 9 \cdot 8 \cdot 1} = 36.$$

$$5.12 \quad 4 \quad [ABD] = 16 \text{ and } [BED] = \frac{1}{4}[ABD].$$

$$5.13 \quad 110^\circ \quad \angle BIC = 90^\circ + \frac{1}{2}\angle A.$$

[A NOTE BEFORE YOU CLOSE IT]

Four things worth remembering

Every IOQM answer is an integer from 00 to 99, so a figure drawn to scale often suggests it. Use that, but do not stop there. A measured angle is a guess; the answer is the one you can prove, with exact values such as $\sqrt{3}$ kept to the end.

Decide the method before you start. The table in Section 5.5 sums up this part in eight rows: an angle, a length in a right triangle, a length from ratios, three given sides, a ratio of areas, a circle, a figure full of right angles, and lattice points or shortest paths. If you cannot say which row a question belongs to, draw the figure again, larger, and look for a hidden circle.

Revise this part three times. The first pass, a few weeks out, is slow and complete, with every exercise attempted. The second, a week out, is only the gold cards and the worked examples. The third, on the morning of the paper, is only the dark cards and the common mistakes of Section 5.6, since these are the mistakes most likely to happen in the paper.

Finally, tell us when we are wrong. If a line here does not match what you work out, write to ritesh@iitianforum.com or academics@iitianforum.com with the page and the line. Corrections go into the next edition, and the student who finds one is credited in it.

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