

[OLYMPIAD MATHEMATICS]

IOQM · PRE-RMO

2027

IOQM *Handbook*

CLASS 8 TO 12

OLYMPIAD

THEORY AND PROBLEMS

IN THIS HANDBOOK

PAPER	IOQM MATHEMATICS
THIS VOLUME	PART IV · ALGEBRA
EACH CHAPTER	THEORY · EXAMPLES · EXERCISES

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IOQM Mathematics Handbook

Part IV · Algebra

FOR IOQM 2027 | EDITION 1 | 2026

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[IOQM · PART IV]

Preface

IOQM is the first stage of the Indian mathematical olympiad programme, and for most students it is the first paper in which a formula is not enough. The questions are short and every answer is a whole number from 00 to 99, but almost every question needs an idea. This handbook is written to build those ideas, one chapter at a time.

The handbook follows the paper. It has four parts, one each for number theory, combinatorics, geometry and algebra, and every part is tied to the IOQM 2026 paper, so that you can see where each idea was actually tested. This volume is Part IV, Algebra. Only four questions of IOQM 2026 were algebra questions, but algebra is used inside many questions of the other three parts: an identity, a sum, a bound.

Everything here stays within school mathematics. Nothing needs more than the algebra of Classes 10 to 12, together with a short introduction to complex numbers in Chapter 2, and every result that is used is either proved in the text or stated with its conditions. The examples and exercises come from the IOQM 2026 paper, from olympiads of many countries, and from our own classes. Every solution has been written afresh, in the way we teach it, and every answer has been checked by computer.

This book is meant to be worked through, not only read. Keep a notebook beside it, attempt every example before you read its solution, and come back later to the questions you could not do.

If something is not clear, if you find a mistake, or if you have a suggestion, write to me at ritesh@iitianforum.com or to our academic team at academics@iitianforum.com. Your message will help every student who uses this book after you.

Ritesh Raj

Founder, IITian Forum · IIT Delhi

[IOQM · PART IV]

How to use this handbook

This page is for you, the student. Part IV has five chapters, in the order in which they are taught. Chapter 1 covers identities and algebraic manipulation. Chapter 2 covers polynomials, and Chapter 3 equations, systems and functional equations. Chapter 4 is about sequences and sums. Chapter 5 covers inequalities, shows how to choose the right method, and ends with a mixed problem set. The answers to all exercises, each with a one-line hint, are at the end of the part. This part can be read on its own; only one example uses $\varphi(n)$ from Part I.

HOW TO STUDY A CHAPTER

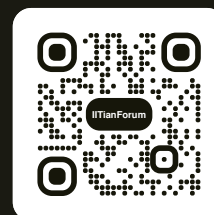
- 01** Read the theory slowly, once. Write out each short proof in your own notebook; that is how a result becomes yours.
- 02** Before reading a worked example, cover its solution and try the question for five minutes. Then read the solution and compare it with yours.
- 03** Learn the gold cards by heart. They hold the definitions, theorems and key ideas that the paper uses again and again.
- 04** Read the dark cards twice. Each one marks a point where students often lose marks.
- 05** Attempt every exercise before you look at the answers. Read the one-line hint only when you are stuck, then finish the question yourself.
- 06** Keep a list of the questions you got wrong, and try them again a week later without looking at the solution.

Questions from IOQM 2026 are marked with their question number, and questions from other olympiads with their source. Stuck on a question, or found a mistake? Write to ritesh@iitianforum.com or academics@iitianforum.com.

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SCAN FOR THE COURSE

[IOQM · PART IV]

Notation and conventions

Each symbol is defined, or first used, in the section shown.

SYMBOL	MEANING	SECTION
s, p, t_n	$x + y, xy$ and $x^n + y^n$, for two numbers x, y	1.3
$\deg P$	the degree of the polynomial P	2.1
$\sum \alpha, \sum \alpha\beta$	the sum of the roots; the sum of their products in pairs	2.2
e_k, p_k	the sum of the products of the roots k at a time; the sum of their k th powers	2.3
$i, \bar{z}, z $	$i^2 = -1$; the conjugate $p - qi$ and the absolute value $\sqrt{p^2 + q^2}$ of $z = p + qi$	2.8
ω	a cube root of unity, $\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$	2.8
$\binom{n}{k}, \sum_{3 k}$	$n! / (k!(n-k)!)$; a sum over the k divisible by 3	2.8
D	$b^2 - 4ac$, the discriminant of $ax^2 + bx + c$	3.2
$ x $	the absolute value of a real number: x if $x \geq 0$, and $-x$ if $x < 0$	3.4
$\lfloor x \rfloor, \{x\}$	the greatest integer $\leq x$; the fractional part $x - \lfloor x \rfloor$	3.5
$\mathbb{R}, x \mapsto g(x)$	the set of real numbers; the map that sends x to $g(x)$	3.7
a_n, S_n	the n th term of a sequence; the sum of its first n terms	4.1
d, r	the common difference of an AP; the common ratio of a GP	4.1
$\varphi(n)$	the number of integers in $1, 2, \dots, n$ coprime to n (Part I, 3.6)	4.1
F_n	Fibonacci numbers: $F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n$	4.5
$\sqrt{a}, \sqrt[n]{a}$	the non-negative square root and n th root of $a \geq 0$; so $\sqrt{x^2} = x $	5.2
AM, GM, HM, QM	the arithmetic, geometric, harmonic and quadratic means	5.3
$\Rightarrow, \Leftrightarrow$	“implies”; “exactly when” (if and only if)	

CONVENTIONS

- (i) **Numbers.** Letters stand for real numbers unless stated otherwise. Positive means greater than 0; non-negative means ≥ 0 .
- (ii) **Polynomials.** Coefficients are real unless stated. The roots of a polynomial of degree n are its n complex roots, counted with multiplicity, as in Vieta's formulas.
- (iii) **Least and largest values.** A least value needs two

things: a bound, and values at which the bound is reached.

- (iv) **“Find all”.** A complete answer gives every solution and also shows that there are no others.
- (v) **Numbering.** 3.5 is section 5 of Chapter 3. Examples are numbered in the same way, and Exercise 2.5 is question 5 of Exercises 2.
- (vi) **Sources.** “(IOQM 2026, Q21)” is question 21 of the IOQM 2026 paper. The name of another olympiad, as in “(RMO 2014)”, gives the source of that question.

[CONTENTS]

Part IV *algebra*

BEFORE YOU BEGIN

i	Preface	3
ii	How to use this handbook	4
iii	Notation and conventions	5

01 · IDENTITIES AND MANIPULATION

1.1	Standard identities	7
1.2	Factorisation methods	7
1.3	Symmetric expressions in two variables	8
1.4	Surds and nested radicals	9
1.5	Sums of squares	9
1.6	Comparing large numbers	10

02 · POLYNOMIALS

2.1	Division and the remainder theorem	11
2.2	Vieta's formulas	11
2.3	Sums of powers of the roots	12
2.4	Roots and graphs	12
2.5	Integer and rational roots	13
2.6	Comparing polynomials	14
2.7	Helper polynomials and interpolation	14
2.8	Roots of unity	15

03 · EQUATIONS AND FUNCTIONAL EQUATIONS

3.1	Combining linear equations	16
3.2	Quadratic equations	17
3.3	Substitutions	17
3.4	Radicals and absolute values	18
3.5	The floor function	19
3.6	Symmetric and cyclic systems	19
3.7	Functional equations	20

04 · SEQUENCES, SERIES AND RECURSIONS

4.1	Arithmetic and geometric progressions	22
4.2	Sums of powers	23
4.3	Telescoping	23
4.4	Recurrences that repeat	24
4.5	Linear recurrences and Fibonacci	25

05 · INEQUALITIES AND PROBLEM SOLVING

5.1	Squares	26
5.2	AM and GM	27
5.3	The chain of means	27
5.4	Cauchy and Schwarz	28
5.5	Maxima and minima in integers	28
5.6	Choosing the method	29
5.7	IOQM 2026 algebra questions	30
5.8	Common mistakes	30
5.9	A mixed problem set	30
A	Answers to the exercises	31
N	A note before you close it	33

CHAPTER 01 IDENTITIES AND MANIPULATION

Most olympiad algebra is manipulation with a purpose: an expression is rewritten until it becomes a product, a square or a constant. This chapter collects the identities that make such rewriting quick, the standard ways of factorising, the use of $x + y$ and xy for symmetric expressions, surds, the fact that a sum of squares is zero only when every square is zero, and the comparison of large powers. Every later chapter uses them.

1.1 STANDARD IDENTITIES

Learn these by heart:

- (i) $(a \pm b)^2 = a^2 \pm 2ab + b^2$, $a^2 - b^2 = (a - b)(a + b)$.
- (ii) $(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$.
- (iii) $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$.
- (iv) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$.
- (v) $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \dots + b^{n-1})$, and for odd n , $a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + \dots + b^{n-1})$.

THEOREM · SUM OF THREE CUBES

$$\begin{aligned} a^3 + b^3 + c^3 - 3abc \\ = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca), \end{aligned}$$

where the second factor is

$$\begin{aligned} a^2 + b^2 + c^2 - ab - bc - ca \\ = \frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2]. \end{aligned}$$

Two consequences are used again and again.

- (i) If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$. For example, $x - y, y - z, z - x$ add to 0, so $(x - y)^3 + (y - z)^3 + (z - x)^3 = 3(x - y)(y - z)(z - x)$.
- (ii) For real a, b, c : $a^3 + b^3 + c^3 = 3abc$ exactly when $a + b + c = 0$ or $a = b = c$, because the second factor is zero only when all three squares are zero.

EXAMPLE 1.1

Distinct non-zero numbers a, b, c have $a + b + c = 0$. Find the value below. (Australian-Polish 1985)

$$\left(\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b}\right)\left(\frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c}\right)$$

For $a = 1, b = 2, c = -3$, the two brackets are

$$\frac{1}{5} - \frac{1}{2} + 3 = \frac{27}{10}, \quad 5 - 2 + \frac{1}{3} = \frac{10}{3}$$

and their product is 9.

In general, on multiplying out, three products such as $\frac{a}{b-c} \cdot \frac{b-c}{a}$ are 1 each. They give 3.

The other six terms form three pairs. One pair is

$$\frac{b-c}{a} \left(\frac{b}{c-a} + \frac{c}{a-b}\right)$$

Over the denominator $(c-a)(a-b)$, the bracket has numerator

$$b(a-b) + c(c-a) = (c-b)(b+c-a)$$

$b + c = -a$, so this numerator is $2a(b-c)$, and the pair is

$$\frac{2(b-c)^2}{(c-a)(a-b)} = \frac{2(b-c)^3}{(a-b)(b-c)(c-a)}$$

Similarly for the other two pairs ($a \rightarrow b \rightarrow c \rightarrow a$). On adding all three,

$$\frac{2[(a-b)^3 + (b-c)^3 + (c-a)^3]}{(a-b)(b-c)(c-a)}$$

The bases add to 0, so the square bracket is $3(a-b)(b-c)(c-a)$ (theorem on the sum of three cubes, Section 1.1). Hence the six terms give $2 \cdot 3 = 6$.

ANSWER $3 + 6 = 9$

1.2 FACTORISATION METHODS

- (i) **Common factors and grouping:** $x^3 - x^2 - 4x + 4 = x^2(x-1) - 4(x-1) = (x-1)(x-2)(x+2)$.
- (ii) **Add and subtract to make a square:** $x^4 + x^2 + 1 = (x^2 + 1)^2 - x^2 = (x^2 + x + 1)(x^2 - x + 1)$.
- (iii) **Sophie Germain:** $a^4 + 4b^4 = (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2)$.
- (iv) **Treat it as a quadratic in one letter.** It factorises

when its discriminant is a perfect square. For $x^2 - 3xy + 2y^2 + x - y$ the discriminant in x is $(y-1)^2$, which gives $(x-y)(x-2y+1)$.

KEY IDEA · PAIR THE FACTORS

In a product of four linear factors, pair them so that both quadratics have the same $x^2 + kx$:

$$(x+1)(x+4) \cdot (x+2)(x+3) + 1 \\ = (u+4)(u+6) + 1 = (u+5)^2,$$

with $u = x^2 + 5x$. So

$$n(n+1)(n+2)(n+3) + 1 = (n^2 + 3n + 1)^2.$$

EXAMPLE 1.2

Find $\sqrt{(31)(30)(29)(28) + 1}$. (AIME 1989)

For 1, 2, 3, 4, pairing the outer numbers and the inner numbers,

$$1 \cdot 4 = 4, \quad 2 \cdot 3 = 6, \quad 4 \cdot 6 + 1 = 5^2$$

Here $4 = 5 - 1$ and $6 = 5 + 1$. In general,

$$(m-1)(m+1) + 1 = m^2$$

Similarly here (key idea *Pair the factors*),

$$28 \cdot 31 = 868, \quad 29 \cdot 30 = 870$$

These are $869 - 1$ and $869 + 1$, so

$$868 \cdot 870 + 1 = 869^2$$

Check: $n^2 + 3n + 1$ at $n = 28$ is $784 + 84 + 1 = 869$.

ANSWER 869

EXAMPLE 1.3

Non-zero integers a, b, c with $a \neq c$ satisfy

$$\frac{a}{c} = \frac{a^2 + b^2}{c^2 + b^2}.$$

Factorise $a^2 + b^2 + c^2$. (Romania 1999)

For example, $a = 1, b = 2, c = 4$ satisfies the condition, as $\frac{1}{4} = \frac{5}{20}$. Here $a^2 + b^2 + c^2 = 21 = 7 \cdot 3$.

In general, on cross-multiplying,

$$a(c^2 + b^2) = c(a^2 + b^2)$$

$$ac^2 - a^2c - b^2c + ab^2 = 0$$

$$ac(c-a) - b^2(c-a) = 0$$

$$(c-a)(ac-b^2) = 0$$

$a \neq c$, so $b^2 = ac$. Using this twice,

$$a^2 + b^2 + c^2 = a^2 + ac + c^2$$

$$= (a+c)^2 - ac = (a+c)^2 - b^2$$

This is a difference of two squares. In the example, $5^2 - 2^2 = 7 \cdot 3$.

ANSWER $(a+b+c)(a-b+c)$

1.3 SYMMETRIC EXPRESSIONS IN TWO VARIABLES

A polynomial in x, y that does not change when x and y are swapped can always be written in terms of

$$s = x + y, \quad p = xy.$$

(i) $x^2 + y^2 = s^2 - 2p$ and $x^3 + y^3 = s^3 - 3sp$.

(ii) $(x-y)^2 = s^2 - 4p$, which gives $x-y$ up to its sign.

(iii) With $t_n = x^n + y^n$: $t_n = s t_{n-1} - p t_{n-2}$, starting from $t_0 = 2, t_1 = s$.

(iv) x, y are the roots of $z^2 - sz + p = 0$, so they are real exactly when $s^2 \geq 4p$.

(v) The same works for $x + \frac{1}{x}$: put $s = x + \frac{1}{x}, p = 1$.

For instance, if $x + y = 4$ and $xy = 1$, then $t_2 = 16 - 2 = 14, t_3 = 64 - 12 = 52$ and $t_4 = 4 \cdot 52 - 14 = 194$, without finding x and y .

EXAMPLE 1.4

Real numbers a, b satisfy $a^4 + a^2b^2 + b^4 = 900$ and $a^2 + ab + b^2 = 45$. Find $2ab$. (Online Math Open 2012)

Putting $a = 2, b = 1$ in the first expression,

$$16 + 4 + 1 = 21 = 7 \cdot 3$$

with $7 = a^2 + ab + b^2$ and $3 = a^2 - ab + b^2$. This suggests a factorisation.

In general, as a difference of two squares,

$$a^4 + a^2b^2 + b^4 = (a^2 + b^2)^2 - a^2b^2$$

$$= (a^2 + ab + b^2)(a^2 - ab + b^2)$$

So $900 = 45(a^2 - ab + b^2)$, and we have

$$a^2 + ab + b^2 = 45$$

$$a^2 - ab + b^2 = 20$$

On subtracting, $2ab = 25$.

ANSWER 25

1.4 SURDS AND NESTED RADICALS

RULES · SURDS

$\sqrt{a \pm 2\sqrt{b}} = \sqrt{m} \pm \sqrt{n}$ when $m + n = a$, $mn = b$ and $m \geq n \geq 0$.

$$\frac{1}{\sqrt{a} + \sqrt{b}} = \frac{\sqrt{a} - \sqrt{b}}{a - b} \text{ for } a \neq b.$$

For the first rule, square the right side: $m + n \pm 2\sqrt{mn}$. For example $7 + 4\sqrt{3} = 7 + 2\sqrt{12}$, and $4 + 3 = 7$, $4 \cdot 3 = 12$, so $\sqrt{7 + 4\sqrt{3}} = 2 + \sqrt{3}$.

If $a + b\sqrt{2} = c + d\sqrt{2}$ with a, b, c, d rational, then $a = c$ and $b = d$, as $\sqrt{2}$ is irrational. The same holds for any irrational square root.

EXAMPLE 1.5

Find $\frac{1}{1 + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \cdots + \frac{1}{\sqrt{99} + \sqrt{100}}$.

For the first term, multiplying numerator and denominator by $\sqrt{2} - 1$,

$$\frac{1}{1 + \sqrt{2}} = \frac{\sqrt{2} - 1}{2 - 1} = \sqrt{2} - 1$$

Similarly for every term: by the second rule for surds (Section 1.4) with $a = k + 1$, $b = k$, $a - b = 1$,

$$\frac{1}{\sqrt{k} + \sqrt{k+1}} = \sqrt{k+1} - \sqrt{k}$$

So the sum is

$$(\sqrt{2} - 1) + (\sqrt{3} - \sqrt{2}) + \cdots + (\sqrt{100} - \sqrt{99})$$

Each of $\sqrt{2}, \dots, \sqrt{99}$ comes once with $+$ and once with $-$, so only $\sqrt{100} - 1 = 10 - 1$ is left.

ANSWER 9

EXAMPLE 1.6

Evaluate the product below. (AIME 1986)

$$(\sqrt{5} + \sqrt{6} + \sqrt{7})(-\sqrt{5} + \sqrt{6} + \sqrt{7}) \\ \times (\sqrt{5} - \sqrt{6} + \sqrt{7})(\sqrt{5} + \sqrt{6} - \sqrt{7}).$$

Each pair below is of the form $(A + B)(A - B) = A^2 - B^2$.

The first two factors are $(\sqrt{6} + \sqrt{7}) \pm \sqrt{5}$. As $(\sqrt{6} + \sqrt{7})^2 = 13 + 2\sqrt{42}$, their product is

$$(13 + 2\sqrt{42}) - 5 = 8 + 2\sqrt{42}$$

The last two are $\sqrt{5} \mp (\sqrt{6} - \sqrt{7})$. As $(\sqrt{6} - \sqrt{7})^2 = 13 - 2\sqrt{42}$, their product is

$$5 - (13 - 2\sqrt{42}) = 2\sqrt{42} - 8$$

Multiplying $2\sqrt{42} + 8$ and $2\sqrt{42} - 8$,

$$(2\sqrt{42})^2 - 8^2 = 168 - 64 = 104$$

ANSWER 104

1.5 SUMS OF SQUARES

THEOREM · A SUM OF SQUARES

For real numbers, $x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0$, with equality exactly when every $x_i = 0$.

So a single equation can fix several real unknowns: rewrite it as a sum of squares equal to 0. The most used case is the identity of 1.1: $a^2 + b^2 + c^2 - ab - bc - ca$ is half the sum of $(a - b)^2$, $(b - c)^2$, $(c - a)^2$, so it is zero only when $a = b = c$.

WHERE MARKS GO · REAL NUMBERS ONLY

The argument needs real numbers. Over complex numbers $1^2 + i^2 = 0$ with neither term zero, so state that the unknowns are real before using it.

EXAMPLE 1.7

Real numbers x, y satisfy $(x^2 + 1)(y^2 + 1) - 2(x + 1)(y + 1) + 4 = 0$. Find $x + y$.

Putting $x = y = 1$, we get $2 \cdot 2 - 2 \cdot 2 \cdot 2 + 4 = 0$. So $x + y = 2$ is possible. We show it is the only case.

On expanding,

$$(x^2 + 1)(y^2 + 1) = x^2y^2 + x^2 + y^2 + 1$$

$$2(x + 1)(y + 1) = 2xy + 2x + 2y + 2$$

So the left side is

$$x^2y^2 + x^2 + y^2 - 2xy - 2x - 2y + 3$$

Writing $3 = 1 + 1 + 1$ and grouping into squares,

$$(xy - 1)^2 + (x - 1)^2 + (y - 1)^2 = 0$$

Each square is 0 (theorem on a sum of squares, Section 1.5), so $x = 1$, $y = 1$, and then $xy = 1$ also holds.

ANSWER $x + y = 2$

EXAMPLE 1.8

Find all real x with $9^x + 4^x + 1 = 6^x + 3^x + 2^x$. (Ukraine 1997)

Putting $x = 0$, both sides are 3. Putting $x = 1$, we get $14 \neq 11$. We show $x = 0$ is the only answer.

Let $a = 3^x$, $b = 2^x$, $c = 1$. Then $9^x = a^2$, $4^x = b^2$, $1 = c^2$, $6^x = ab$, $3^x = ca$, $2^x = bc$, and the equation becomes

$$a^2 + b^2 + c^2 = ab + bc + ca$$

i.e., by the identity in Section 1.1,

$$\frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2] = 0$$

So $a = b = c$ (theorem on a sum of squares). Hence $3^x = 1$, which gives $x = 0$.

ANSWER $x = 0$

1.6 COMPARING LARGE NUMBERS

- (i) **Match the exponents:** $2^{300} = 8^{100} < 9^{100} = 3^{200}$.
- (ii) **Match the bases:** $4^{50} = 2^{100}$, $8^{40} = 2^{120}$.
- (iii) **Insert a middle term** that is easy to compare with both numbers.
- (iv) **Compare a ratio with 1** when the two numbers are close.

EXAMPLE 1.9

Which is larger, 31^{11} or 17^{14} ?

31 is a little less than $32 = 2^5$ and 17 a little more than $16 = 2^4$, so we insert a power of 2 in the middle (Section 1.6):

$$31^{11} < 32^{11} = 2^{55}$$

$$17^{14} > 16^{14} = 2^{56}$$

$$\text{As } 2^{55} < 2^{56},$$

$$31^{11} < 2^{55} < 2^{56} < 17^{14}$$

ANSWER 17^{14}

WHERE MARKS GO · EACH LINK MUST BE CLEAR

Do not compare by a rough count of digits. Write a chain in which every single inequality is obvious, as in the example above.

EXERCISES 1

- 1.1 Real numbers a, b, c satisfy $a + b + c = 0$ and $abc = 4$. Find $a^3 + b^3 + c^3$.
- 1.2 If $a + b = 5$ and $ab = 3$, find $a^3 + b^3$.
- 1.3 If $x + \frac{1}{x} = 3$, find $x^5 + \frac{1}{x^5}$.
- 1.4 Find $\sqrt{20 \cdot 21 \cdot 22 \cdot 23 + 1}$.
- 1.5 Find $\sqrt{11 + 6\sqrt{2}} + \sqrt{11 - 6\sqrt{2}}$.
- 1.6 Evaluate $(\sqrt{2} + \sqrt{11} + \sqrt{13})(-\sqrt{2} + \sqrt{11} + \sqrt{13}) \times (\sqrt{2} - \sqrt{11} + \sqrt{13})(\sqrt{2} + \sqrt{11} - \sqrt{13})$.
- 1.7 Find the value of $\frac{1^4 + 2007^4 + 2008^4}{1^2 + 2007^2 + 2008^2}$. (British MO 2007)
- 1.8 Find $\sqrt{\frac{11^4 + 100^4 + 111^4}{2}}$. (Japan 2016)
- 1.9 Real numbers x, y satisfy $x^2 + y^2 + 5 = 4x + 2y$. Find $x + y$.
- 1.10 Real numbers x, y satisfy $(2x + 1)^2 + y^2 + (y - 2x)^2 = \frac{1}{3}$. Find $x + y$. (Croatia 2003)
- 1.11 How many positive integers n satisfy $n^{200} < 5^{300}$?
- 1.12 Write $2^{18} + 1$ in the form $a^4 + 4b^4$ and find its largest prime factor.
- 1.13 Real numbers x, y, z , not all equal, satisfy $x^3 + y^3 + z^3 = 3xyz$ and $x + 2y + 3z = 6$. If $x = y$, find z .

CHAPTER 02 POLYNOMIALS

A polynomial question in IOQM usually asks for one number: a remainder, a coefficient, a value $P(k)$ or a symmetric expression in the roots. Very often the roots themselves are never found. This chapter covers division and remainders, the relations between roots and coefficients, sums of powers of the roots, integer roots, comparing two polynomials, helper polynomials with known roots, and roots of unity.

2.1 DIVISION AND THE REMAINDER THEOREM

$P(x) = a_n x^n + \cdots + a_1 x + a_0$ with $a_n \neq 0$ has **degree** n and **leading coefficient** a_n . It is **monic** if $a_n = 1$.

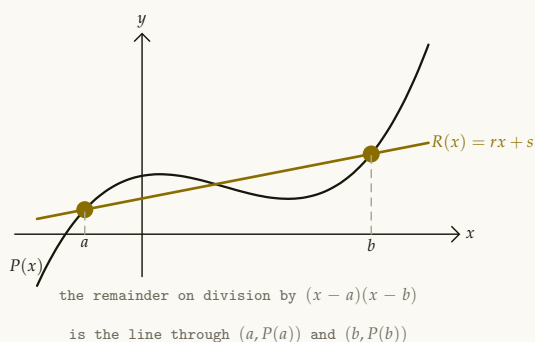
THEOREM · DIVISION

For polynomials P and $D \neq 0$ there are unique polynomials Q and R with

$$P(x) = D(x)Q(x) + R(x),$$

where $R = 0$ or $\deg R < \deg D$.

- (i) **Remainder theorem:** on division by $x - a$ the remainder is $P(a)$.
- (ii) **Factor theorem:** $x - a$ is a factor of $P(x)$ exactly when $P(a) = 0$.
- (iii) On division by $(x - a)(x - b)$ with $a \neq b$, the remainder is $rx + s$, and $ra + s = P(a)$, $rb + s = P(b)$ fix r and s .
- (iv) $P(1)$ is the sum of the coefficients and $P(0) = a_0$.



EXAMPLE 2.1

Find the remainder when x^{100} is divided by $x^2 - 3x + 2$.

$x^2 - 3x + 2 = (x - 1)(x - 2)$. Let the remainder be $rx + s$ (degree less than 2).

$$x^{100} = (x - 1)(x - 2)Q(x) + rx + s$$

Putting $x = 1$ and $x = 2$,

$$r + s = 1, \quad 2r + s = 2^{100}$$

as $(x - 1)(x - 2) = 0$ there. On subtracting, $r = 2^{100} - 1$, so $s = 1 - r = 2 - 2^{100}$.

ANSWER $(2^{100} - 1)x + 2 - 2^{100}$

2.2 VIETA'S FORMULAS

THEOREM · VIETA

If $ax^2 + bx + c$ has roots α, β , then

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

If $ax^3 + bx^2 + cx + d$ has roots α, β, γ , then

$$\sum \alpha = -\frac{b}{a}, \quad \sum \alpha\beta = \frac{c}{a}, \quad \alpha\beta\gamma = -\frac{d}{a}.$$

Here $\sum \alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha$. In general, for degree n the sum of the products of the roots taken k at a time is $(-1)^k a_{n-k} / a_n$. For example $x^3 - 6x^2 + 11x - 6$ has roots 1, 2, 3: their sum is 6, the sum of products in pairs is 11 and the product is 6.

Any symmetric polynomial in the roots can be found from these sums, so it is fixed by the coefficients without solving the equation. Also, a root satisfies the equation: if $t^2 - 4t + 1 = 0$ then $t^2 = 4t - 1$, and higher powers of t come down in the same way.

EXAMPLE 2.2

a, b are the roots of $x^2 + 3x + 1 = 0$ and c, d are the roots of $x^2 - 4x + 1 = 0$. Find $(a + c)(b + c)(a - d)(b - d)$. (Singapore Junior 2008)

By Vieta,

$$a + b = -3, \quad ab = 1$$

$$c + d = 4, \quad cd = 1$$

Also $c^2 = 4c - 1$, as c is a root of $x^2 - 4x + 1 = 0$.

$$(a + c)(b + c) = c^2 + (a + b)c + ab = c^2 - 3c + 1$$

which is $4c - 1 - 3c + 1 = c$. Similarly, with $d^2 = 4d - 1$,

$$(a - d)(b - d) = d^2 - (a + b)d + ab = d^2 + 3d + 1$$

which is $7d$. Hence the product is $c \cdot 7d = 7cd = 7$.

ANSWER 7

EXAMPLE 2.3

r is a positive integer such that the three roots of $x^3 - 30x^2 + rx - 780 = 0$ are the sides of a right-angled triangle. Find the sum of the squares of the digits of r . (Australian MO 2011)

Let the legs be a, b and the hypotenuse c . By Vieta,

$$a + b + c = 30, \quad abc = 780$$

Also $a^2 + b^2 = c^2$ (Pythagoras theorem). Using $a^2 + b^2 = (a + b)^2 - 2ab$ and $a + b = 30 - c$,

$$(30 - c)^2 - 2ab = c^2$$

On cancelling c^2 , $900 - 60c = 2ab$, so $ab = 450 - 30c$. Substituting in $c \cdot ab = 780$,

$$c(450 - 30c) = 780, \quad c^2 - 15c + 26 = 0$$

On solving, $c = 2$ or $c = 13$.

$c = 2$: $a + b = 28$, $ab = 390$; these a, b are not real, as $28^2 < 4 \cdot 390$.

$c = 13$: $a + b = 17$, $ab = 60$, so the legs are 5 and 12. Check: $5^2 + 12^2 = 13^2$ and $5 \cdot 12 \cdot 13 = 780$.

Again by Vieta, r is the sum of the products in pairs:

$$r = ab + c(a + b) = 60 + 13 \cdot 17 = 281$$

ANSWER $2^2 + 8^2 + 1^2 = 69$

WHERE MARKS GO · SYMMETRIC OR NOT

$\alpha - \beta$ is not symmetric. Find $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$ from Vieta, and then decide the sign from the question.

2.3 SUMS OF POWERS OF THE ROOTS

Let α, β, γ be the roots of $x^3 - e_1x^2 + e_2x - e_3$, so $e_1 = \sum \alpha, e_2 = \sum \alpha\beta, e_3 = \alpha\beta\gamma$. Write $p_k = \alpha^k + \beta^k + \gamma^k$.

THEOREM · NEWTON'S IDENTITIES

$$p_1 = e_1, \quad p_2 = e_1p_1 - 2e_2,$$

$$p_3 = e_1p_2 - e_2p_1 + 3e_3,$$

$$p_k = e_1p_{k-1} - e_2p_{k-2} + e_3p_{k-3} \quad (k \geq 4).$$

The last line comes from the equation itself: each root satisfies $\alpha^3 = e_1\alpha^2 - e_2\alpha + e_3$. Multiply by α^{k-3} and add over the three roots. For two numbers the same idea gives the rule of 1.3, $t_n = st_{n-1} - pt_{n-2}$.

EXAMPLE 2.4

$a + b + c = 1$, $a^2 + b^2 + c^2 = 2$ and $a^3 + b^3 + c^3 = 3$. Find $a^4 + b^4 + c^4$.

Here $p_1 = 1, p_2 = 2, p_3 = 3$. By Newton's identities, $e_1 = p_1 = 1$, and from $p_2 = e_1p_1 - 2e_2$,

$$2 = 1 - 2e_2, \quad e_2 = -\frac{1}{2}$$

From $p_3 = e_1p_2 - e_2p_1 + 3e_3$,

$$3 = 2 + \frac{1}{2} + 3e_3, \quad e_3 = \frac{1}{6}$$

Then the identity with $k = 4$ gives

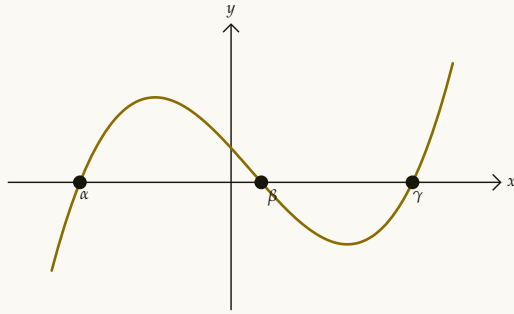
$$p_4 = e_1p_3 - e_2p_2 + e_3p_1 = 3 + 1 + \frac{1}{6}$$

ANSWER $\frac{25}{6}$

2.4 ROOTS AND GRAPHS

- (i) A polynomial of degree $n \geq 1$ has exactly n complex roots, counted with multiplicity. So it has at most n real roots.
- (ii) With real coefficients, non-real roots come in conjugate pairs $p \pm qi$. A polynomial of odd degree therefore has at least one real root.
- (iii) If $P(a)$ and $P(b)$ have opposite signs, P has a root between a and b .
- (iv) At a double root the graph touches the x axis without crossing it.
- (v) **Descartes:** the number of positive roots, counted with multiplicity, is at most the number of sign

changes in the list of non-zero coefficients, and differs from it by an even number.



three distinct real roots exactly when the local maximum is above the axis and the local minimum below it

EXAMPLE 2.5

a, b are positive real numbers and the equation $ax^2 - 23x + b = x^3$ has exactly two distinct real solutions, whose sum is 12. Find $a + b$. (Cuba 2004)

On rearranging,

$$x^3 - ax^2 + 23x - b = 0$$

Non-real roots come in a conjugate pair (Section 2.4); such a pair would leave only one real root, but there are two. So all three roots are real, two of them equal. Let the roots be r, r, s ($r \neq s$). The solutions are r, s , so $r + s = 12$. By Vieta,

$$2r + s = a, \quad r^2 + 2rs = 23, \quad r^2s = b$$

Putting $s = 12 - r$ in the middle one,

$$r^2 + 2r(12 - r) = 23$$

i.e., $r^2 - 24r + 23 = 0$, or $(r - 1)(r - 23) = 0$.

r	s	a	b
1	11	13	11
23	-11	35	-5819

$r = 1$, as $b > 0$. Check:

$$(x - 1)^2(x - 11) = x^3 - 13x^2 + 23x - 11$$

ANSWER $a + b = 13 + 11 = 24$

2.5 INTEGER AND RATIONAL ROOTS

THEOREM · RATIONAL ROOTS

If $\frac{p}{q}$, in lowest terms, is a root of a polynomial with integer coefficients, then $p \mid a_0$ and $q \mid a_n$. For a monic polynomial every rational root is an integer dividing a_0 .

For example, the rational roots of $2x^3 - 3x^2 - 3x + 2$ can only be $\pm 1, \pm 2, \pm \frac{1}{2}$. Testing them gives the roots 2, -1 and $\frac{1}{2}$.

KEY IDEA · DIFFERENCES OF VALUES

If P has integer coefficients and a, b are integers, then

$$a - b \mid P(a) - P(b),$$

because $a - b$ divides every $a^k - b^k$.

So P has no integer root if $P(0), P(1)$ are both odd.

EXAMPLE 2.6

A polynomial P with integer coefficients has $P(17) = 10$ and $P(24) = 17$. The equation $P(n) = n + 3$ has two distinct integer solutions n_1, n_2 . Find $n_1 n_2$. (AIME 2005)

Let $Q(x) = P(x) - x - 3$. The solutions of $P(n) = n + 3$ are the integer roots of Q .

$$Q(17) = 10 - 20 = -10$$

$$Q(24) = 17 - 27 = -10$$

So $Q(x) + 10$ has the roots 17 and 24. By the factor theorem,

$$Q(x) + 10 = (x - 17)(x - 24)R(x)$$

and R has integer coefficients, as the divisor is monic.

At a solution n , $Q(n) = 0$, so

$$(n - 17)(n - 24)R(n) = 10$$

$\therefore (n-17)(n-24) \mid 10$. Let $u = n-17$; then $n-24 = u-7$.

u	$u(u-7)$
0, 7	0
1, 6	-6
2, 5	-10
3, 4	-12
-1, 8	8

For $u \leq -2$ or $u \geq 9$, the two factors have the same sign and $u(u-7) \geq 2 \cdot 9 = 18 > 10$. In the table only -10 divides 10. Hence $u = 2$ or 5, i.e., $n = 19$ or 22.

ANSWER $n_1 n_2 = 19 \cdot 22 = 418$

2.6 COMPARING POLYNOMIALS

- (i) If two polynomials are equal for all x , their coefficients are equal.
- (ii) Two polynomials of degree at most n that agree at $n+1$ points are the same polynomial.
- (iii) $\deg(PQ) = \deg P + \deg Q$, and $\deg P(Q(x)) = \deg P \cdot \deg Q$ when Q is not constant.
- (iv) Compare the degrees first, then the leading coefficients, then the rest. Putting $x = 0, \pm 1$ or a known root gives quick equations.

EXAMPLE 2.7

Find the number of non-constant polynomials $P(x)$ with real coefficients such that $P(x^2) = P(P(x))$. (IOQM 2026, Q11)

Let $\deg P = n \geq 1$. Comparing degrees (Section 2.6), $P(x^2)$ has degree $2n$ and $P(P(x))$ has degree n^2 , so

$$2n = n^2, \quad n = 2$$

$\therefore P(x) = ax^2 + bx + c$ with $a \neq 0$, and $P(x^2) = ax^4 + bx^2 + c$.

x^4 terms: $P(P(x)) = aP(x)^2 + bP(x) + c$ has leading term $a \cdot a^2 x^4$. So $a^3 = a$, and $a^2 = 1$ (as $a \neq 0$).

x^3 terms: $P(x^2)$ has none. In $aP(x)^2$ it is $a \cdot 2abx^3 = 2bx^3$, and $bP(x) + c$ has degree 2. So $b = 0$.

Now $P(x) = ax^2 + c$. Using $a^2 = 1$,

$$\begin{aligned} P(P(x)) &= a(ax^2 + c)^2 + c \\ &= ax^4 + 2cx^2 + ac^2 + c \end{aligned}$$

Comparing x^2 terms with $P(x^2) = ax^4 + c$, $2c = 0$, so $c = 0$.

Hence $P(x) = x^2$ or $-x^2$. Both satisfy the equation: for $P(x) = -x^2$, $P(P(x)) = -(-x^2)^2 = -x^4 = P(x^2)$.

ANSWER 2

EXAMPLE 2.8

$P(x)$ is a quadratic polynomial with integer coefficients such that

$$P(P(x)) = P(x)^2 + x^2 + x + 100.$$

Find $P(4)$. (CMIMC 2017)

Let $P(x) = ax^2 + bx + c$ and $y = P(x)$. Then

$$ay^2 + by + c = y^2 + x^2 + x + 100$$

x^4 terms: y^2 has leading term $a^2 x^4$, so $a \cdot a^2 = a^2$, i.e., $a = 1$ (as $a \neq 0$).

Now y^2 cancels from both sides. Putting $y = x^2 + bx + c$ in $by + c$,

$$bx^2 + b^2x + bc + c = x^2 + x + 100$$

The x^2 terms give $b = 1$ (and then $b^2 = 1$ for the x terms). The constants give $c + c = 100$, so $c = 50$.

Hence $P(x) = x^2 + x + 50$. Check: $P(y) = y^2 + y + 50$, and $y + 50 = x^2 + x + 100$.

ANSWER $P(4) = 16 + 4 + 50 = 70$

WHERE MARKS GO · SUBSTITUTE BACK

Comparing coefficients gives conditions that are necessary. Put the polynomials you found back into the equation, and read the question again for words such as “non-constant” or “integer coefficients”.

2.7 HELPER POLYNOMIALS AND INTERPOLATION

Through $n+1$ points with distinct x values there is exactly one polynomial of degree at most n . When P takes known values at many points, build a new polynomial that is zero at those points.

KEY IDEA · BUILD A HELPER WITH KNOWN ROOTS

If $P(k) = f(k)$ for $k = 1, 2, \dots, m$, then $Q(x) = P(x) - f(x)$, or a multiple such as $xP(x) - 1$ when $f(k) = \frac{1}{k}$, has the roots $1, \dots, m$. So

$$Q(x) = (x-1)(x-2)\cdots(x-m)S(x),$$

and the degree tells you what S can be.

EXAMPLE 2.9

$P(x) = x^4 + ax^3 + bx^2 + cx + d$ with $P(1) = 10$, $P(2) = 20$, $P(3) = 30$. Find $\frac{P(12) + P(-8)}{10}$. (ARML 1984)

$P(x) - 10x$ is 0 at $x = 1, 2, 3$ and is monic of degree 4, so by the factor theorem

$$P(x) - 10x = (x-1)(x-2)(x-3)(x-r)$$

for some number r , which we do not need to find.

At $x = 12$, the first three factors give $11 \cdot 10 \cdot 9 = 990$:

$$P(12) = 120 + 990(12 - r)$$

At $x = -8$, they give $(-9)(-10)(-11) = -990$, and $x - r = -(8 + r)$:

$$P(-8) = -80 + 990(8 + r)$$

On adding, the terms with r cancel, as $(12 - r) + (8 + r) = 20$:

$$P(12) + P(-8) = 40 + 990 \cdot 20 = 19840$$

ANSWER $\frac{19840}{10} = 1984$

EXAMPLE 2.10

P has degree 4 and $P(k) = \frac{k}{k+1}$ for $k = 0, 1, 2, 3, 4$. Find $P(5)$.

Let $Q(x) = (x+1)P(x) - x$. For $k = 0, 1, 2, 3, 4$,

$$Q(k) = (k+1) \cdot \frac{k}{k+1} - k = 0$$

Q has degree 5 and these five roots, so

$$Q(x) = cx(x-1)(x-2)(x-3)(x-4)$$

At $x = -1$, the factor $x + 1$ is 0, so $Q(-1) = 1$, and the right side is

$$c(-1)(-2)(-3)(-4)(-5) = -120c$$

$\therefore c = -\frac{1}{120}$. At $x = 5$, the right side is $120c = -1$, so

$$6P(5) - 5 = -1, \quad P(5) = \frac{4}{6}$$

ANSWER $\frac{2}{3}$

The differences of the values of a polynomial of degree n become constant after n rounds; the n th difference is $n! a_n$. For a cubic with $P(1), \dots, P(4)$ equal to 1, 2, 4, 8, the differences are 1, 2, 4, then 1, 2, then 1. Keeping the third difference 1, the next second difference is 3, the next first difference is 7, and $P(5) = 8 + 7 = 15$.

2.8 ROOTS OF UNITY

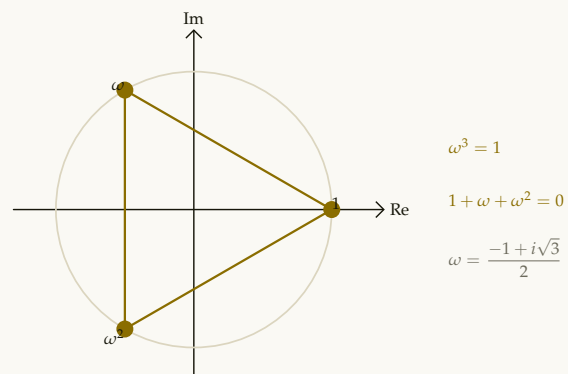
A complex number is $z = p + qi$ with $i^2 = -1$; its conjugate is $\bar{z} = p - qi$ and its absolute value is $|z| = \sqrt{p^2 + q^2}$. The equation $x^n = 1$ has n roots, the n th roots of unity

$$1, \omega, \omega^2, \dots, \omega^{n-1}, \quad \omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}.$$

They are equally spaced on the circle $|z| = 1$, and for $n \geq 2$ they add to 0.

RULES · CUBE ROOTS OF UNITY

$$\begin{aligned} \omega^3 &= 1, & 1 + \omega + \omega^2 &= 0, & \bar{\omega} &= \omega^2, \\ x^2 + x + 1 &= (x - \omega)(x - \omega^2), & 1 + \omega &= -\omega^2. \end{aligned}$$



So $x^2 + x + 1$ divides a polynomial P with real coefficients exactly when $P(\omega) = 0$; then $P(\omega^2) = 0$ as well, being its conjugate. Powers of ω depend only on the exponent modulo 3.

EXAMPLE 2.11

For how many integers n with $1 \leq n \leq 100$ does $x^2 + x + 1$ divide $x^{2n} + x^n + 1$?

Let $P(x) = x^{2n} + x^n + 1$. By the rules for cube roots of unity, $x^2 + x + 1$ divides P exactly when $P(\omega) = 0$. Powers of ω depend only on the remainder of n on division by 3:

n	ω^n, ω^{2n}	$P(\omega)$
$3m$	1, 1	3
$3m+1$	ω, ω^2	0
$3m+2$	ω^2, ω	0

Hence the condition holds exactly when $3 \nmid n$. Among $1, \dots, 100$ there are 33 multiples of 3.

ANSWER $100 - 33 = 67$

The same substitution counts binomial coefficients. Adding $(1+x)^n$ at $x = 1, \omega, \omega^2$ keeps only the terms with $3 \mid k$:

$$\sum_{3 \mid k} \binom{n}{k} = \frac{2^n + (1+\omega)^n + (1+\omega^2)^n}{3}.$$

EXERCISES 2

- 2.1 $P(x)$ leaves remainder 4 on division by $x-1$ and remainder -5 on division by $x+2$. Find the remainder on division by $(x-1)(x+2)$.

- 2.2 The four roots of $x^4 + kx^2 + 90x - 2009 = 0$ are x_1, x_2, x_3, x_4 , and $x_1x_2 = 49$. Find k . (Singapore Junior 2009)
- 2.3 The roots of $x^3 - 12x^2 + kx - 48 = 0$ are in arithmetic progression. Find k .
- 2.4 α, β, γ are the roots of $2x^3 - 5x^2 + 7x - 3 = 0$. Find $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$.
- 2.5 α, β, γ are the roots of $x^3 - x - 1 = 0$. Find $\alpha^5 + \beta^5 + \gamma^5$.
- 2.6 Integers $a < b < c$ satisfy $a + b + c = 95$ and $a^2 + b^2 + c^2 = 3083$. Find c . (CMIMC 2016)
- 2.7 The equation $4x^4 - 12x^3 - 7x^2 + 22x + 14 = 0$ has four distinct real roots $p > q > r > s$, and two of the roots add up to 1. Find $\lfloor 10(p+q) \rfloor$.
- 2.8 Find the sum of the rational roots of $2x^3 - 5x^2 - 6x + 4 = 0$.
- 2.9 P has integer coefficients and $P(a) = P(b) = P(c) = P(d) = 5$ for four distinct integers a, b, c, d . For how many integers k is $P(k) = 8$?
- 2.10 How many polynomials P of degree at most 2, with real coefficients, satisfy $P(x)^2 = P(x^2)$?
- 2.11 P is monic of degree 4 with $P(1) = 3, P(2) = 6, P(3) = 9, P(4) = 12$. Find $P(5)$.
- 2.12 Find the remainder when $x^{2026} + x^{1013} + 1$ is divided by $x^2 + x + 1$.
- 2.13 Find $\sum_{3 \mid k} \binom{12}{k} = \binom{12}{0} + \binom{12}{3} + \dots + \binom{12}{12}$.

CHAPTER 03 EQUATIONS AND FUNCTIONAL EQUATIONS

An equation in IOQM is rarely solved by a formula alone. Usually the equations are combined, a block is replaced by a new letter, or the question is turned into a sum of squares or a statement about $x+y$ and xy . This chapter covers linear and quadratic equations, substitutions, radicals and absolute values, the floor function, symmetric and cyclic systems, and functional equations.

3.1 COMBINING LINEAR EQUATIONS

When there are more unknowns than equations, the unknowns are not fixed, but the question asks for a combination that is. Look for multipliers λ, μ such that $\lambda \cdot (\text{first}) + \mu \cdot (\text{second})$ is exactly the combination

asked for.

EXAMPLE 3.1

3 shirts, 7 trousers and 1 wallet cost Rs. 307. 4 shirts, 10 trousers and 1 wallet cost Rs. 411. Find the cost of 1 shirt, 1 trouser and 1 wallet together. (AIMO 2015)

Let the prices be s, t, w :

$$3s + 7t + w = 307, \quad 4s + 10t + w = 411$$

Two equations cannot determine three prices: $(s, t, w) = (14, 30, 55)$ and $(20, 28, 51)$ both satisfy them. But both give $s + t + w = 99$.

Multiply the equations by λ, μ and add. Comparing the coefficients of s, t, w with $s + t + w$,

$$3\lambda + 4\mu = 1, \quad 7\lambda + 10\mu = 1, \quad \lambda + \mu = 1$$

From the first and last, $\lambda = 3, \mu = -2$. These satisfy the middle one also ($21 - 20 = 1$). Therefore

$$s + t + w = 3 \cdot 307 - 2 \cdot 411 = 921 - 822$$

ANSWER Rs. 99

3.2 QUADRATIC EQUATIONS

THEOREM · COMPLETING THE SQUARE

With $D = b^2 - 4ac$ and $a \neq 0$,

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}.$$

- (i) Two real roots if $D > 0$, one double root if $D = 0$, no real root if $D < 0$.
- (ii) With integer coefficients, the roots are rational exactly when D is a perfect square.
- (iii) For $a > 0$ the least value is $-\frac{D}{4a}$, taken at $x = -\frac{b}{2a}$.
- (iv) For $f(x) = ax^2 + bx + c$, both roots exceed k exactly when $D \geq 0$, $a f(k) > 0$ and $-\frac{b}{2a} > k$.

EXAMPLE 3.2

Find the real numbers x, y with $(4x^2 + 6x + 4)(4y^2 - 12y + 25) = 28$. (USA-MTS 1998)

On completing the square in each bracket (theorem in Section 3.2),

$$4x^2 + 6x + 4 = 4\left(x + \frac{3}{4}\right)^2 + \frac{7}{4}$$

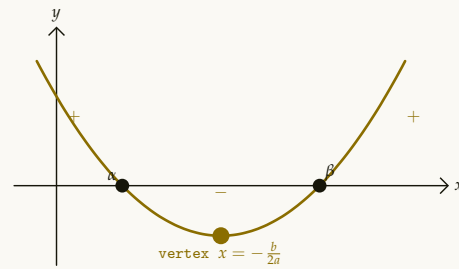
$$4y^2 - 12y + 25 = (2y - 3)^2 + 16$$

The product is at least 28. A square is never negative, so the brackets are $\geq \frac{7}{4}$ and ≥ 16 . Both are positive, so the product is $\geq \frac{7}{4} \cdot 16 = 28$.

Equality. The product is exactly 28, so each bracket takes its least value:

$$x + \frac{3}{4} = 0, \quad 2y - 3 = 0$$

ANSWER $x = -\frac{3}{4}, y = \frac{3}{2}$



$a > 0$: negative exactly between the roots, least at the vertex

3.3 SUBSTITUTIONS

- (i) A block that appears more than once becomes a new letter.
- (ii) **Reciprocal equations** (the same coefficients from both ends): divide by x^2 and put $y = x + \frac{1}{x}$, so $x^2 + \frac{1}{x^2} = y^2 - 2$.
- (iii) Exponentials: put $a = 2^x, b = 3^x$; then $4^x = a^2$ and $6^x = ab$.
- (iv) Symmetry about $x = m$: put $x = m + t$.

EXAMPLE 3.3

Solve $x^2(2 - x)^2 = 1 + 2(1 - x)^2$ for real x . (Bangladesh 2018)

On replacing x by $2 - x$, the equation does not change. So it is symmetric about $x = 1$, and we write everything in terms of $1 - x$:

$$x(2 - x) = 1 - (1 - x)^2$$

(Check at $x = 3$: $3 \cdot (-1) = -3$ and $1 - 4 = -3$.)

Putting $t = (1 - x)^2$, we get

$$(1 - t)^2 = 1 + 2t, \quad t^2 - 4t = 0$$

So $t = 0$ or $t = 4$.

$t = 0$: $1 - x = 0$, so $x = 1$.

$t = 4$: $1 - x = \pm 2$, so $x = -1$ or $x = 3$.

ANSWER $x = -1, 1, 3$

EXAMPLE 3.4

Find the sum of the real roots of $x^4 - 5x^3 + 6x^2 - 5x + 1 = 0$.

The coefficients 1, -5, 6, -5, 1 are the same in reverse order, so this is a reciprocal equation (Section 3.3).

$x = 0$ is not a root. On dividing by x^2 and grouping,

$$\left(x^2 + \frac{1}{x^2}\right) - 5\left(x + \frac{1}{x}\right) + 6 = 0$$

Putting $y = x + \frac{1}{x}$, so that $x^2 + \frac{1}{x^2} = y^2 - 2$,

$$y^2 - 5y + 4 = 0, \quad y = 1 \text{ or } y = 4$$

Multiplying $x + \frac{1}{x} = y$ by x ,

y	equation	D
1	$x^2 - x + 1 = 0$	-3
4	$x^2 - 4x + 1 = 0$	12

Only $y = 4$ gives real roots ($D > 0$). They are $2 \pm \sqrt{3}$, with sum

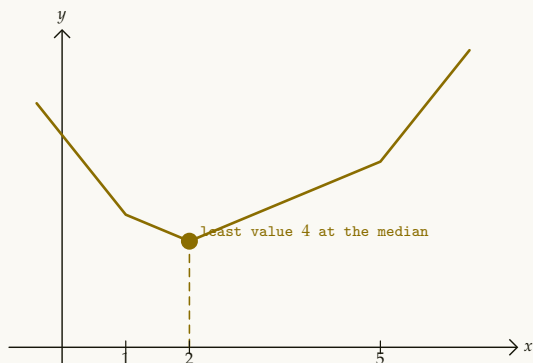
$$(2 + \sqrt{3}) + (2 - \sqrt{3}) = 4$$

ANSWER 4

3.4 RADICALS AND ABSOLUTE VALUES

- (i) $\sqrt{A} = B$ needs $B \geq 0$. Squaring keeps every solution but can add new ones, so every answer must be checked in the original equation.
- (ii) $|A| = B$ exactly when $B \geq 0$ and $A = \pm B$.
- (iii) With $a_1 \leq a_2 \leq \dots \leq a_n$, the sum $|x - a_1| + \dots + |x - a_n|$ is a broken line. It is least at the middle number when n is odd, and at every x between the two middle numbers when n is even.

For example, $\sqrt{x+5} = x-1$ gives $x^2 - 3x - 4 = 0$, so $x = 4$ or $x = -1$. Only $x = 4$ has $x-1 \geq 0$, so it is the only solution.



$|x-1| + |x-2| + |x-5|$ is piecewise linear, least at the middle point

EXAMPLE 3.5

Find the least value of $|x-1| + |x-2| + |x-5| + |x-9|$.

For a few values of x :

x	0	3	5	7
sum	17	11	11	15

The least value appears to be 11.

At least 11. Pairing the outer terms and the inner terms, and using $|u| + |v| \geq |u+v|$,

$$|x-1| + |9-x| \geq |(x-1) + (9-x)| = 8$$

$$|x-2| + |5-x| \geq |(x-2) + (5-x)| = 3$$

On adding, the sum is $\geq 8+3=11$.

Equality when $2 \leq x \leq 5$. Then $x-1, 9-x, x-2, 5-x$ are all ≥ 0 , so both lines are equalities.

ANSWER 11

EXAMPLE 3.6

Solve in real numbers: $\sqrt{x^2 - y} = z - 1$, $\sqrt{y^2 - z} = x - 1$, $\sqrt{z^2 - x} = y - 1$. (Morocco 2015)

A square root is never negative, so $z-1, x-1, y-1$ are all ≥ 0 :

$$x \geq 1, \quad y \geq 1, \quad z \geq 1$$

On squaring each equation,

$$x^2 - y = (z-1)^2,$$

$$y^2 - z = (x-1)^2,$$

$$z^2 - x = (y-1)^2$$

Let $S = x^2 + y^2 + z^2$ and $T = x + y + z$. On adding the three, the left side is $S - T$ and the right side is $S - 2T + 3$:

$$S - T = S - 2T + 3, \quad T = 3$$

Each of x, y, z is ≥ 1 and their sum is 3. If one were greater than 1, the sum would exceed 3. So $x = y = z = 1$.

Check: each equation becomes $\sqrt{1-1} = 1-1$, i.e., $0 = 0$.

ANSWER (1, 1, 1)

WHERE MARKS GO · CHECK AFTER SQUARING

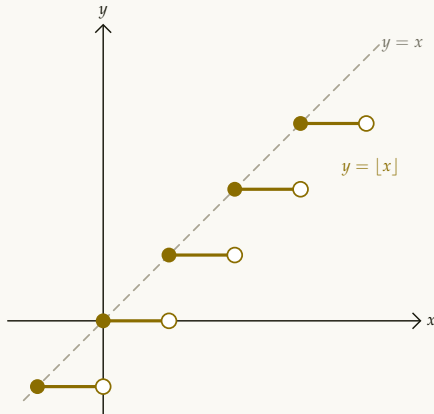
After squaring, put each answer back into the original equation. A root that makes a square root equal to a negative number is not a solution.

3.5 THE FLOOR FUNCTION

DEFINITION · FLOOR AND FRACTIONAL PART

$\lfloor x \rfloor$ is the greatest integer $\leq x$ and $\{x\} = x - \lfloor x \rfloor$.
So $x = n + f$ with $n = \lfloor x \rfloor$ an integer and $0 \leq f < 1$, and

$$\lfloor x \rfloor = n \iff n \leq x < n + 1.$$



$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1; \text{ write } x = n + f \text{ with } 0 \leq f < 1$$

- (i) $\lfloor x + m \rfloor = \lfloor x \rfloor + m$ for every integer m .
- (ii) **Hermite:** $\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = \lfloor 2x \rfloor$.
- (iii) **Method:** write $n = \lfloor x \rfloor$, solve for x in terms of n , and then use $n \leq x < n + 1$ to find the possible n .

EXAMPLE 3.7

How many real numbers x satisfy $x^2 - 8\lfloor x \rfloor + 7 = 0$?

Putting $x = 7$, we get $49 - 8 \cdot 7 + 7 = 0$. So $x = 7$ is one solution.

In general, let $n = \lfloor x \rfloor$ (method of Section 3.5). Then

$$x^2 = 8n - 7$$

$x^2 \geq 0$, so $8n - 7 \geq 0$, and $n \geq 1$ as n is an integer. Then $x \geq n \geq 1$, so $x = \sqrt{8n - 7}$.

On squaring $n \leq x$ (both sides positive),

$$n^2 \leq 8n - 7, \quad (n - 1)(n - 7) \leq 0$$

So $1 \leq n \leq 7$. Checking each n ,

n	1	2	3	4	5	6	7
x^2	1	9	17	25	33	41	49
$\lfloor x \rfloor$	1	3	4	5	5	6	7

$\lfloor x \rfloor = n$ only for $n = 1, 5, 6, 7$. Hence

$$x = 1, \sqrt{33}, \sqrt{41}, 7$$

ANSWER 4

3.6 SYMMETRIC AND CYCLIC SYSTEMS

- (i) Subtract two equations and take out a factor such as $x - y$. Then treat the cases $x = y$ and $x \neq y$ separately.
- (ii) Add the equations, or add them in pairs, to look for squares.
- (iii) With two unknowns use $s = x + y$, $p = xy$ (1.3). Real x, y need $s^2 \geq 4p$.
- (iv) With three unknowns use $x + y + z$, $xy + yz + zx$ and xyz : then x, y, z are the roots of $t^3 - (x + y + z)t^2 + (xy + yz + zx)t - xyz$.

EXAMPLE 3.8

Find all real (x, y, z) with

$$\begin{aligned}x^2 - yz + xy + zx &= 82, \\y^2 - zx + xy + yz &= -18, \\z^2 - xy + yz + zx &= 18.\end{aligned}$$

(COMC 2005)

On adding the first two equations ($-yz$ and yz cancel, and so do zx and $-zx$),

$$x^2 + 2xy + y^2 = 64, \quad (x + y)^2 = 64$$

Similarly, the other two pairs give

$$(y + z)^2 = 0, \quad (z + x)^2 = 100$$

So $y = -z$. Then $-yz = z^2$ and $xy + zx = x(y + z) = 0$, so the first equation becomes

$$x^2 + z^2 = 82$$

Using $(x + z)^2 = x^2 + z^2 + 2xz$, the sum and product of x and z are

$$s = x + z = \pm 10, \quad p = xz = \frac{100 - 82}{2} = 9$$

Sum 10, product 9 give 1 and 9; sum -10 gives -1 and -9 . With $y = -z$, we get four triples.

These also satisfy $(x + y)^2 = 64$:

$$(x + y)^2 = (x - z)^2 = s^2 - 4p = 100 - 36$$

ANSWER $\pm(9, -1, 1)$ and $\pm(1, -9, 9)$

EXAMPLE 3.9

Complex numbers x, y, z satisfy

$$\begin{aligned}x^2 + y^2 + z &= xy, \\x + y^2 + z^2 &= yz, \\x^2 + y + z^2 &= xz.\end{aligned}$$

Find the sum of all distinct possible values of

$$|(x^2 - y)(y^2 - z)(z^2 - x)|.$$

(IOQM 2026, Q21)

Subtracting the second equation from the first,

$$(x^2 - z^2) - (x - z) - y(x - z) = 0$$

On taking out $x - z$, and similarly for the other two pairs,

$$\begin{aligned}(x - z)(x + z - 1 - y) &= 0, \\(y - x)(y + x - 1 - z) &= 0, \\(y - z)(y + z - 1 - x) &= 0\end{aligned}$$

All different. Then the second factors are 0:

$$x + z = 1 + y, \quad x + y = 1 + z$$

On adding, $2x = 2$, so $x = 1$. Similarly $y = z = 1$, which is not possible as they are different.

All equal, $x = y = z = t$. The first equation gives $t^2 + t = 0$. $t = 0$ gives the value 0; $t = -1$ makes each bracket $1 + 1 = 2$, so the value is 8.

Exactly two equal, say $x = y \neq z$. As $y \neq z$, the third line gives $y + z - 1 - x = 0$, so $z = 1$. The first equation becomes $2x^2 + 1 = x^2$, so $x = \pm i$. Both $(i, i, 1)$ and $(-i, -i, 1)$ satisfy all three equations.

For $(i, i, 1)$ the brackets are $-1 - i, -2, 1 - i$, and

$$(-1 - i)(1 - i) = -2, \quad |(-2)(-2)| = 4$$

$(-i, -i, 1)$ also gives 4. The system does not change under $x \rightarrow y \rightarrow z \rightarrow x$, so $y = z \neq x$ and $z = x \neq y$ also give 4.

The values are 0, 4, 8.

ANSWER $0 + 4 + 8 = 12$

WHERE MARKS GO · NEVER DIVIDE BY ZERO

From $(x - z)(x + z - 1 - y) = 0$ you may not cancel $x - z$. Write both cases. In IOQM 2026, Q21, the values 0 and 8 come only from the case $x = y = z$.

3.7 FUNCTIONAL EQUATIONS

A functional equation asks for a function, or for one value of it, from a rule that holds for all x (and y).

- (i) Put special values: $x = 0, y = 0, x = y, y = -x, x = 1$.
- (ii) Replace x by $1 - x$ or $\frac{1}{x}$ to get a second equation, and solve the two as a linear system.
- (iii) The maps $x \mapsto \frac{1}{1-x} \mapsto \frac{x-1}{x} \mapsto x$ come back after three rounds; $x \mapsto 1 - x$ and $x \mapsto \frac{1}{x}$ come back after two.
- (iv) Check every function you find in the original equation.

EXAMPLE 3.10

f is defined on the rationals other than 0, 1 and

$$x^2 f(x) + f\left(\frac{x-1}{x}\right) = 2x^2.$$

Find $f\left(\frac{1}{2}\right)$. (Singapore Senior 2011)

On applying $x \mapsto \frac{x-1}{x}$ again and again from $x = \frac{1}{2}$ (e.g. $\frac{1/2-1}{1/2} = -1$),

$$\frac{1}{2} \rightarrow -1 \rightarrow 2 \rightarrow \frac{1}{2}$$

i.e., back after three rounds (Section 3.7).

Let $a = f\left(\frac{1}{2}\right)$, $b = f(-1)$, $c = f(2)$. Putting $x = \frac{1}{2}, -1, 2$ in the equation,

$$\frac{1}{4}a + b = \frac{1}{2},$$

$$b + c = 2,$$

$$4c + a = 8$$

From the first, $b = \frac{1}{2} - \frac{1}{4}a$.

From the second, $c = 2 - b = \frac{3}{2} + \frac{1}{4}a$, so $4c = 6 + a$.

Substituting in the third,

$$6 + a + a = 8, \quad a = 1$$

ANSWER $f\left(\frac{1}{2}\right) = 1$

EXAMPLE 3.11

Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ with

$$f((x-y)^2) = x^2 - 2xyf(x) + f(y)^2.$$

For a prime p , let q be the sum of all possible values of $f(p)$. Find the largest two-digit prime q that can arise. (Australia 2001)

For $f(x) = x$, the right side is $x^2 - 2xy + y^2 = (x-y)^2$, the left side. So $f(x) = x$ is a solution. We now find all such f .

Putting $x = y = 0$, we get $f(0) = f(0)^2$, so $f(0) = 0$ or 1. Putting $y = x$,

$$f(0) = x^2 - 2xf(x) + f(x)^2 = (f(x) - x)^2$$

Case $f(0) = 0$. Then $f(x) = x$ for every x .

Case $f(0) = 1$. Then $f(x) = x + 1$ or $x - 1$ for each x . Putting $y = 0$, we get $f(x^2) = x^2 + 1$; putting $x = 0$, we get $f(y^2) = -2y + f(y)^2$. So

$$f(y)^2 = y^2 + 1 + 2y = (y+1)^2$$

If $f(y) = y - 1$, then $(y-1)^2 = (y+1)^2$, so $y = 0$; but $f(0) = 1$, not -1 . So $f(y) = y + 1$ for every y . It satisfies the equation: $x^2 - 2y(x+1) + (y+1)^2 = (x-y)^2 + 1$.

So $f(p)$ is p or $p+1$, and $q = 2p+1$. Checking the largest two-digit primes q ,

q	$p = \frac{q-1}{2}$	p prime?
97	48	no
89	44	no
83	41	yes

ANSWER $q = 83$

EXERCISES 3

- 3.1 a, b, c are distinct positive integers and $N = 5a + 3b + 5c = 4a + 5b + 4c$, where $131 < N < 150$. Find $a + b + c$. (Cayley 1998)
- 3.2 Blocks of one colour all weigh the same whole number of kilograms, and different colours weigh different amounts. Each of the collections (5 red, 3 blue, 5 green), (4 red, 5 blue, 4 green) and (7 red, 4 blue, k green) weighs w kg, where $30 < w < 50$. If N is the weight of 6 red, 7 blue and 3 green blocks, find $k + w + N$. (AIMO 2014)
- 3.3 The fraction $\frac{a}{b}$ in lowest terms equals $\frac{2}{n} + \frac{1}{n^2}$ for a positive integer n , and $a + b = 1024$. Find a . (Australian Intermediate 2015)
- 3.4 How many pairs of real numbers (a, b) satisfy $2ab = 6(a+b) - 13$ and $a^2 + b^2 = 4$? (Bosnia and Herzegovina 2018)
- 3.5 How many pairs of real numbers (a, b) satisfy $a^2 + b^2 = 25$ and $3(a+b) - ab = 15$? (Germany 2016)
- 3.6 Find all real x with $2^x + 3^x - 4^x + 6^x - 9^x = 1$. (Korea 2000)
- 3.7 Real numbers $x, y, z > 1$ satisfy
- $$x + y + z + \frac{3}{x-1} + \frac{3}{y-1} + \frac{3}{z-1} = 2(\sqrt{x+2} + \sqrt{y+2} + \sqrt{z+2}).$$
- Find x . (Nordic 1992)
- 3.8 How many integers x satisfy $|x-2| + |x-7| = 5$?
- 3.9 Find the least real x with $\lfloor x \rfloor + \lfloor 2x \rfloor = 10$.

- 3.10 Find all real (x, y, z) with $(x-1)(y-1)(z-1) = xyz - 1$ and $(x-2)(y-2)(z-2) = xyz - 2$. (Metropolises 2018)
- 3.11 Non-zero reals a, b, c satisfy $a^2 + a = b^2$, $b^2 + b = c^2$, $c^2 + c = a^2$. Find $(a-b)(b-c)(c-a)$. (Poland 2018)

- 3.12 f is defined for real $x \neq 0, 1$ and $f(x) + \frac{1}{2x}f\left(\frac{1}{1-x}\right) = 1$. Find $12f(12)$. (Australian MO 2003)
- 3.13 $f(x) + 2f(1-x) = x^2$ for all real x . Find $f(3)$.

CHAPTER 04 SEQUENCES, SERIES AND RECURSIONS

A sequence question gives a rule and asks for a far term or a long sum. Writing out the first few terms is always the first thing to do: the pattern, a period or a telescoping sum usually shows itself early. This chapter covers progressions, sums of powers, telescoping, recurrences that repeat, and linear recurrences such as the Fibonacci numbers.

4.1 ARITHMETIC AND GEOMETRIC PROGRESSIONS

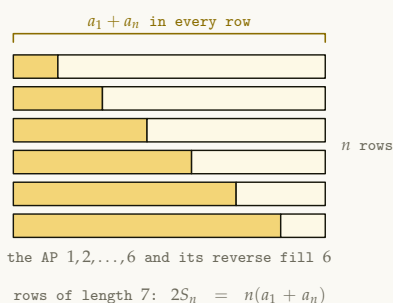
An **AP** has a constant difference d , so $a_n = a_1 + (n-1)d$. A **GP** has a constant ratio r , so $a_n = a_1 r^{n-1}$.

THEOREM · SUMS

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(2a_1 + (n-1)d)$$

for an AP, and for a GP with $r \neq 1$

$$S_n = a_1 \frac{r^n - 1}{r - 1}, \quad S_\infty = \frac{a_1}{1 - r} \quad (|r| < 1).$$



- (i) a, b, c are in AP when $2b = a + c$, in GP when $b^2 = ac$ (all non-zero), and in HP when $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP.
- (ii) $(m+1) + \dots + n = \frac{(n-m)(n+m+1)}{2}$, a sum of consecutive integers.
- (iii) **Arithmetico-geometric sums:** multiply by r and subtract. For $S = \sum_{k \geq 1} \frac{k}{2^k}$, $S - \frac{S}{2} = \sum_{k \geq 1} \frac{1}{2^k} = 1$, so $S = 2$.

EXAMPLE 4.1

How many 3-term geometric progressions a, b, c of positive integers are there with $a < b < c$ and $c = 8000$? (Math Prize for Girls 2018)

The ratio need not be an integer. For ratio $\frac{5}{4}$, we get

$$5120, 6400, 8000$$

$$\text{and } 6400^2 = 5120 \cdot 8000.$$

In general, let the ratio be $\frac{p}{q}$ in lowest terms, $p > q \geq 1$. As $c = 8000$,

$$b = \frac{8000q}{p}, \quad a = \frac{8000q^2}{p^2}$$

$\gcd(p^2, q^2) = 1$, so a is an integer if and only if $p^2 \mid 8000$. Then $p \mid 8000$ too, so b is an integer.

$8000 = 2^6 \cdot 5^3$, so $p^2 \mid 8000$ gives $p = 2^i 5^j$, $i \leq 3$, $j \leq 1$. For each such $p > 1$, q is any integer with $1 \leq q < p$ and $\gcd(p, q) = 1$:

p	allowed q	count
2	1	1
4	odd	2
8	odd	4
5	1, 2, 3, 4	4
10	odd, $5 \nmid q$	4
20	odd, $5 \nmid q$	8
40	odd, $5 \nmid q$	16

Each pair (p, q) gives a different ratio, hence a different progression.

ANSWER $1 + 2 + 4 + 4 + 4 + 8 + 16 = 39$

4.2 SUMS OF POWERS

THEOREM · POWER SUMS

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$



- (i) $1 \cdot 2 + 2 \cdot 3 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}$.
- (ii) $\sum_{k=1}^n k(k+1)(k+2) = \frac{1}{4} n(n+1)(n+2)(n+3)$.
- (iii) $1 + 3 + 5 + \dots + (2n-1) = n^2$.
- (iv) A sum of a polynomial in k is split into these standard sums.

EXAMPLE 4.2

Find $1^2 + 3^2 + 5^2 + \dots + 39^2$.

The odd numbers $1, 3, 5, \dots, 39$ are $2k-1$ for $k = 1, 2, \dots, 20$, and

$$(2k-1)^2 = 4k^2 - 4k + 1$$

By the power sums of Section 4.2,

$$\sum_{k=1}^{20} k^2 = \frac{20 \cdot 21 \cdot 41}{6} = 2870$$

$$\sum_{k=1}^{20} k = \frac{20 \cdot 21}{2} = 210$$

$\therefore$ the required sum is

$$4 \cdot 2870 - 4 \cdot 210 + 20 = 11480 - 840 + 20$$

Check with $k = 1, 2, 3$:

$$1 + 9 + 25 = 35 = 4 \cdot 14 - 4 \cdot 6 + 3$$

ANSWER 10660

4.3 TELESOPING

KEY IDEA · WRITE EACH TERM AS A DIFFERENCE

If $t_k = b_k - b_{k+1}$, then

$$t_1 + t_2 + \dots + t_n = b_1 - b_{n+1},$$

because every middle term cancels. A product of $\frac{b_k}{b_{k+1}}$ collapses in the same way to $\frac{b_1}{b_{n+1}}$.

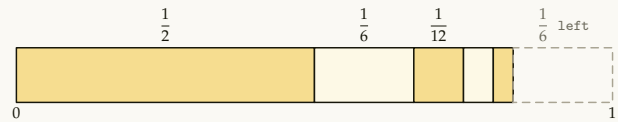
$$(i) \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

$$(ii) \frac{2}{k(k+1)(k+2)} = \frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}.$$

$$(iii) k \cdot k! = (k+1)! - k!, \text{ so } \sum_{k=1}^n k \cdot k! = (n+1)! - 1.$$

$$(iv) 1 - \frac{1}{k^2} = \frac{k-1}{k} \cdot \frac{k+1}{k}, \text{ which telescopes as a product.}$$

$$(v) \text{ A recurrence of the form } b_n - b_{n-1} = c_n \text{ gives } b_n = b_0 + c_1 + \dots + c_n.$$



$$1/(k(k+1)) = 1/k - 1/(k+1): \text{ five pieces fill } 1 - 1/6$$

EXAMPLE 4.3

$a_1 = a_2 = 2$ and, for $n \geq 2$,

$$\frac{2a_{n-1}a_n}{a_{n-1}a_{n+1} - a_n^2} = n^3 - n.$$

Find the integer closest to $\sum_{n=1}^{50} \frac{a_{n+1}}{a_n}$. (Singapore 2012)

Let $r_n = \frac{a_{n+1}}{a_n}$, so $r_1 = 1$. Putting $n = 2$,

$$\frac{2 \cdot 2 \cdot 2}{2a_3 - 2^2} = 2^3 - 2 = 6$$

$\therefore 2a_3 - 4 = \frac{4}{3}, a_3 = \frac{8}{3}$ and $r_2 = \frac{4}{3}$.

In general, on dividing the numerator and denominator of the left side by $a_{n-1}a_n$,

$$\frac{2}{r_n - r_{n-1}} = n^3 - n = (n-1)n(n+1)$$

So $r_n - r_{n-1} = \frac{2}{(n-1)n(n+1)}$, and by the second identity in Section 4.3 (with $k = n-1$),

$$r_n - r_{n-1} = \frac{1}{(n-1)n} - \frac{1}{n(n+1)}$$

On adding these for the indices $2, 3, \dots, n$, the middle terms cancel:

$$r_n = 1 + \frac{1}{2} - \frac{1}{n(n+1)} = \frac{3}{2} - \frac{1}{n(n+1)}$$

Check: $n = 1$ gives $\frac{3}{2} - \frac{1}{2} = 1 = r_1$; $n = 2$ gives $\frac{3}{2} - \frac{1}{6} = \frac{4}{3} = r_2$.

In $r_1 + \dots + r_{50}$, the 50 copies of $\frac{3}{2}$ give 75, and by the first identity in Section 4.3,

$$\sum_{n=1}^{50} \frac{1}{n(n+1)} = 1 - \frac{1}{51}$$

$\therefore$ the required sum is $75 - 1 + \frac{1}{51} = 74 + \frac{1}{51}$.

ANSWER 74

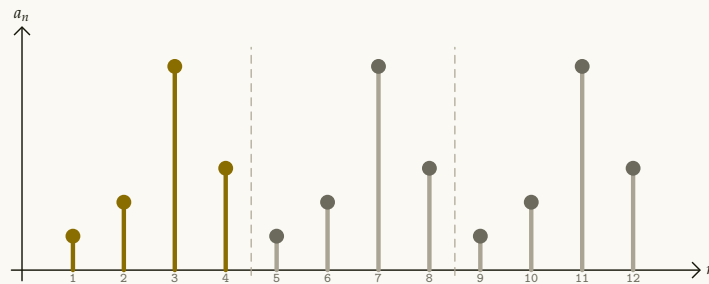
4.4 RECURRENCES THAT REPEAT

Many rules give a sequence that repeats. Compute terms until something repeats or a pattern is clear, and then prove the pattern from the rule. Some periods to know, when every term is defined:

(i) $x_{n+1} = \frac{1}{1-x_n}$ has period 3, as in 3.7.

(ii) $x_{n+1} = \frac{1+x_n}{1-x_n}$ has period 4, and $a_{n+1} = a_n - a_{n-1}$ has period 6.

(iii) $a_{n+1}a_{n-1} = a_n$ (non-zero terms) has period 6:
 $a, b, \frac{b}{a}, \frac{1}{a}, \frac{1}{b}, \frac{a}{b}, a, \dots$



IOQM 2026, Q16: $a_{n+4} = a_n$, so 1, 2, 6, 3 repeats

EXAMPLE 4.4

A sequence of real numbers satisfies

$$\frac{a_{n+3} - a_{n+2}}{a_n - a_{n+1}} = \frac{a_{n+3} + a_{n+2}}{a_n + a_{n+1}} \quad (n \geq 1),$$

and $a_{55} = 6, a_{66} = 2, a_{77} = 1$. If $N = a_1^2 + a_2^2 + \dots + a_{2026}^2$, find the sum of the digits of N . (IOQM 2026, Q16)

On cross-multiplying,

$$\begin{aligned} (a_{n+3} - a_{n+2})(a_n + a_{n+1}) \\ = (a_{n+3} + a_{n+2})(a_n - a_{n+1}) \end{aligned}$$

On expanding, $a_{n+3}a_n$ and $-a_{n+2}a_{n+1}$ cancel from both sides, leaving $2a_{n+3}a_{n+1} = 2a_{n+2}a_n$. So

$$a_{n+3}a_{n+1} = a_{n+2}a_n \quad (n \geq 1)$$

No term after a_1 is 0. If $a_k = 0$ for some $k \geq 2$, then putting $n = k-1$ above, $a_{k+2}a_k = a_{k+1}a_{k-1}$,

so $a_{k+1}a_{k-1} = 0$. Then two neighbouring terms are 0, and their sum is a denominator $a_n + a_{n+1}$ of the given rule, which is not possible.

Replacing n by $n+1$,

$$a_{n+4}a_{n+2} = a_{n+3}a_{n+1} = a_{n+2}a_n$$

$a_{n+2} \neq 0$, so $a_{n+4} = a_n$, i.e., the sequence repeats every 4 terms.

given	index	so
$a_{77} = 1$	$77 = 4 \cdot 19 + 1$	$a_1 = 1$
$a_{66} = 2$	$66 = 4 \cdot 16 + 2$	$a_2 = 2$
$a_{55} = 6$	$55 = 4 \cdot 13 + 3$	$a_3 = 6$

Putting $n = 1$, $a_4a_2 = a_3a_1$, so $2a_4 = 6$, $a_4 = 3$. Hence the sequence is 1, 2, 6, 3 repeated. Check ($n = 1$ in the given rule): $\frac{3-6}{1-2} = 3 = \frac{3+6}{1+2}$.

One block of squares adds to $1 + 4 + 36 + 9 = 50$, and $2026 = 4 \cdot 506 + 2$, so N has 506 blocks and then $a_1^2 + a_2^2$:

$$N = 506 \cdot 50 + 1 + 4 = 25305$$

$$\text{ANSWER } 2 + 5 + 3 + 0 + 5 = 15$$

EXAMPLE 4.5

$a_1 = 3$, $a_2 = -1$ and $a_n a_{n-2} + a_{n-1} = 2$ for $n \geq 3$. Find $a_1 + a_2 + \dots + a_{99}$. (Cuba 2006)

Here $a_n = \frac{2-a_{n-1}}{a_{n-2}}$, and the first terms are

$$3, -1, 1, -1, 3, 1, \frac{1}{3}, \frac{5}{3}, 1, \frac{3}{5}, \frac{7}{5}, 1, \dots$$

The sequence does not repeat. But from a_3 onwards it is in blocks of three:

$$(1, -1, 3), (1, \frac{1}{3}, \frac{5}{3}), (1, \frac{3}{5}, \frac{7}{5}), \dots$$

Each block is $1, x, 2-x$.

This continues for ever. Let $a_{n-1} = 1$. Then

$$\begin{aligned} a_n &= \frac{2-1}{a_{n-2}} = \frac{1}{a_{n-2}} \\ a_{n+1} &= \frac{2-a_n}{1} = 2-a_n \\ a_{n+2} &= \frac{2-a_{n+1}}{a_n} = \frac{a_n}{a_n} = 1 \end{aligned}$$

No denominator is 0: x starts at -1 , and if $x < 1$ then $2-x > 1$, so the next $x = \frac{1}{2-x}$ (first line) lies between 0 and 1.

So each block adds to $1 + x + (2-x) = 3$. Now $a_1 + a_2 = 2$, the 96 terms $a_3, \dots, a_{98}$ form 32 blocks, and $a_{99} = 1$ starts the next block. $\therefore$ the required sum is

$$2 + 32 \cdot 3 + 1 = 99$$

ANSWER 99

WHERE MARKS GO · CANCEL ONLY NON-ZERO TERMS

To go from $a_{n+4}a_{n+2} = a_{n+2}a_n$ to $a_{n+4} = a_n$ you must know $a_{n+2} \neq 0$. Give the reason in one line, as in the IOQM example.

4.5 LINEAR RECURRENCES AND FIBONACCI

THEOREM · LINEAR RECURRENCE

Let $a_n = p a_{n-1} + q a_{n-2}$, and let α, β be the roots of $x^2 = px + q$. If $\alpha \neq \beta$, then $a_n = A\alpha^n + B\beta^n$; if $\alpha = \beta$, then $a_n = (A + Bn)\alpha^n$. The first two terms fix A and B .

It also works the other way: $S_n = A\alpha^n + B\beta^n$ always satisfies

$$S_{n+2} = (\alpha + \beta) S_{n+1} - \alpha\beta S_n.$$

This is the rule $t_n = s t_{n-1} - p t_{n-2}$ of 1.3, now with weights A and B .

EXAMPLE 4.6

$ax + by = 3$, $ax^2 + by^2 = 7$, $ax^3 + by^3 = 16$, $ax^4 + by^4 = 42$. Find $ax^5 + by^5$. (AIME 1990)

Let $S_n = ax^n + by^n$, so $S_1, \dots, S_4$ are 3, 7, 16, 42, and let

$$s = x + y, \quad p = xy$$

Then x, y are the roots of $t^2 = st - p$, so by the rule in Section 4.5,

$$S_{n+2} = s S_{n+1} - p S_n$$

Putting $n = 1$ and $n = 2$,

$$16 = 7s - 3p, \quad 42 = 16s - 7p$$

Multiplying the first by 7, the second by 3 and subtracting, $s = 112 - 126$. Substituting $s = -14$ in the first, $-98 - 3p = 16$. Hence

$$s = -14, \quad p = -38$$

Putting $n = 3$,

$$S_5 = sS_4 - pS_3 = -14 \cdot 42 + 38 \cdot 16$$

i.e., $S_5 = -588 + 608$.

ANSWER 20

The **Fibonacci numbers** are $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... With $F_0 = 0$:

- (i) **Binet:** $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$, $\alpha = \frac{1 + \sqrt{5}}{2}$, $\beta = \frac{1 - \sqrt{5}}{2}$.
 (ii) $F_1 + F_2 + \dots + F_n = F_{n+2} - 1$ and $F_1^2 + \dots + F_n^2 = F_n F_{n+1}$.
 (iii) $F_{n+1} F_{n-1} - F_n^2 = (-1)^n$.
 (iv) If $x^2 = x + 1$, then $x^n = F_n x + F_{n-1}$ for $n \geq 1$.

EXERCISES 4

- 4.1 The 5th term of an AP is 11 and the 12th term is 32. Find the sum of the first 20 terms.
 4.2 The integers from 1 to n add to N and those from 1 to m add to M , where $n > m + 1$ and $N - M = 2012$. Find $n + m$. (Crux Mathematicorum)

4.3 Find $\sum_{k=1}^{\infty} \frac{k}{3^k}$.

4.4 For which n is $1^3 + 2^3 + \dots + n^3 = 2025$?

4.5 Find $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \dots \left(1 - \frac{1}{100^2}\right)$.

4.6 $a_0 = a_1 = 1$ and $a_{n+1} = \frac{a_{n-1}}{1 + n a_{n-1} a_n}$ for $n \geq 1$.

Find $\frac{1}{a_{10} a_{11}}$. (Singapore Senior 2009)

4.7 Find p/q for the two sums below. (Hungary-Israel 1990)

$$p = \frac{1989}{2} - \frac{1988}{3} + \frac{1987}{4} - \dots + \frac{1}{1990},$$

$$q = \frac{1}{996} + \frac{3}{997} + \frac{5}{998} + \dots + \frac{1989}{1990}.$$

4.8 Positive reals satisfy $a_1 + a_2 = 45$ and $a_{n-1} a_{n+1} = a_n$ for $n \geq 2$. Find $a_{2023} + a_{2024}$. (Australian MO 2010)

4.9 $x_1 = 2$ and $x_{n+1} = \frac{1}{1 - x_n}$. Find x_{2026} .

4.10 $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$, and $f_7 = 83$. Find $f_1 + f_2 + \dots + f_{10}$. (Math Prize for Girls 2012)

4.11 Numbers $x_1, \dots, x_{10}$ satisfy $x_i + 2x_{i+1} = 1$ for $i = 1, \dots, 9$. Find $x_1 + 512x_{10}$. (Math Prize for Girls 2014)

4.12 Find $F_1^2 + F_2^2 + \dots + F_{10}^2$.

4.13 $a_1 = 1$, $a_2 = 5$ and $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 3$. Find a_{10} .

CHAPTER 05

INEQUALITIES AND PROBLEM SOLVING

In IOQM an inequality almost always appears as a question about a largest or least value. The answer has two parts: a bound, and values at which the bound is reached. This chapter covers sums of squares, AM and GM, the chain of means, Cauchy and Schwarz, and maxima and minima over the integers. It ends with a table for choosing the method, the four algebra questions of IOQM 2026, a list of common mistakes and a mixed problem set.

5.1 SQUARES

The basic inequalities below all come from $x^2 \geq 0$:

- (i) $a^2 + b^2 \geq 2ab$, that is $(a - b)^2 \geq 0$.
 (ii) $x + \frac{1}{x} \geq 2$ for $x > 0$, with equality only at $x = 1$.
 (iii) $a^2 + b^2 + c^2 \geq ab + bc + ca$ (1.1).
 (iv) $(a + b + c)^2 \geq 3(ab + bc + ca)$, which is the same

inequality.

To find a minimum, write the expression as a constant plus squares, as in 1.5: for example $x^2 + y^2 - 4x + 6y = (x - 2)^2 + (y + 3)^2 - 13$, least value -13 at $(2, -3)$.

EXAMPLE 5.1

Find all real x, y with $x^2 + 2y^2 + \frac{1}{2} \leq x(2y + 1)$. (RMO 2014)

On multiplying by 2 and taking all terms to the left,

$$2x^2 + 4y^2 + 1 - 4xy - 2x \leq 0$$

Writing $2x^2 = x^2 + x^2$,

$$(x^2 - 4xy + 4y^2) + (x^2 - 2x + 1) \leq 0$$

$$(x - 2y)^2 + (x - 1)^2 \leq 0$$

A square is never negative, so both squares are 0:
 $x = 1$ and $x = 2y$, i.e., $y = \frac{1}{2}$.

Check: L.H.S. $= 1 + \frac{1}{2} + \frac{1}{2} = 2$, R.H.S. $= 1 \cdot 2 = 2$.

ANSWER $x = 1, y = \frac{1}{2}$

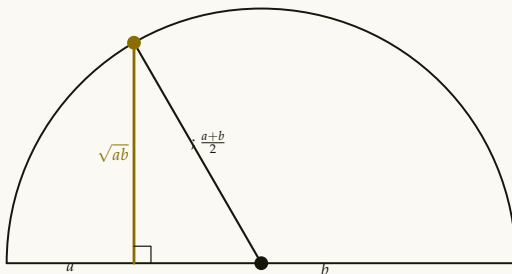
5.2 AM AND GM

THEOREM · AM AND GM

For non-negative $a_1, \dots, a_n$,

$$\frac{a_1 + a_2 + \dots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \dots a_n},$$

with equality exactly when all the a_i are equal.



the half chord never exceeds the radius:

$$\sqrt{ab} \leq \frac{a+b}{2}, \text{ equal only when } a = b$$

- (i) With the sum fixed, the product is largest when the terms are equal. With the product fixed, the sum is least when the terms are equal.
- (ii) **Split terms so that the product is constant.** For $x > 0$,

$$x^2 + \frac{2}{x} = x^2 + \frac{1}{x} + \frac{1}{x} \geq 3\sqrt[3]{1} = 3,$$

with equality when $x^2 = \frac{1}{x}$, i.e. $x = 1$.

- (iii) Decide the equality case first. The split is right only if all the parts can be equal at the same time.

EXAMPLE 5.2

Positive reals x, y satisfy $x + y = 1$. Find the least value of $\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right)$.

For $x = \frac{1}{4}, y = \frac{3}{4}$, the value is $5 \cdot \frac{7}{3} = \frac{35}{3}$.

On expanding,

$$\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) = 1 + \frac{2}{xy}$$

as $\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} = \frac{1}{xy}$.

By AM and GM, $\sqrt{xy} \leq \frac{x+y}{2} = \frac{1}{2}$, so $xy \leq \frac{1}{4}$.

At least 9. $\frac{2}{xy} \geq 8$, so the value is $\geq 1 + 8 = 9$.

Equality when $x = y = \frac{1}{2}$. The value is $3 \cdot 3 = 9$.

ANSWER 9

EXAMPLE 5.3

Positive reals a, b, c satisfy $a + b + c = 6$. Find the largest value of ab^2c^3 .

For $a = b = c = 2$, $ab^2c^3 = 2 \cdot 4 \cdot 8 = 64$.

Splitting b into two equal parts and c into three, the six parts add to 6:

$$a + \frac{b}{2} + \frac{b}{2} + \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = 6$$

Their product is

$$a \cdot \frac{b^2}{4} \cdot \frac{c^3}{27} = \frac{ab^2c^3}{108}$$

At most 108. The six parts have AM $\frac{6}{6} = 1$, so by AM and GM their product is $\leq 1^6 = 1$, i.e., $ab^2c^3 \leq 108$.

Equality when $a = 1, b = 2, c = 3$. Equality needs the six parts equal, each 1: $a = 1, \frac{b}{2} = 1, \frac{c}{3} = 1$. Then $ab^2c^3 = 1 \cdot 4 \cdot 27 = 108$.

ANSWER 108

5.3 THE CHAIN OF MEANS

THEOREM · HM, GM, AM, QM

For positive $a_1, \dots, a_n$,

$$\frac{n}{\sum \frac{1}{a_i}} \leq \sqrt[n]{\prod a_i} \leq \frac{\sum a_i}{n} \leq \sqrt{\frac{\sum a_i^2}{n}},$$

and each inequality is an equality exactly when all the a_i are equal.



always $HM \leq GM \leq AM \leq QM$ for positive numbers, equal only when they agree

Two consequences are used most:

$$(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 9,$$

$$a^2 + b^2 + c^2 \geq \frac{(a + b + c)^2}{3}.$$

The second holds for all real a, b, c , being the case $n = 3$ of $n \sum a_i^2 \geq (\sum a_i)^2$.

EXAMPLE 5.4

Real numbers a, b, c, d, e satisfy $a + b + c + d + e = 8$ and $a^2 + b^2 + c^2 + d^2 + e^2 = 16$. Find the largest possible value of e . (USAMO 1978)

For the other four numbers,

$$a + b + c + d = 8 - e$$

$$a^2 + b^2 + c^2 + d^2 = 16 - e^2$$

By $n \sum a_i^2 \geq (\sum a_i)^2$ (Section 5.3) with $n = 4$,

$$(8 - e)^2 \leq 4(16 - e^2)$$

$$64 - 16e + e^2 \leq 64 - 4e^2$$

$$e(5e - 16) \leq 0$$

At most $\frac{16}{5}$. So $0 \leq e \leq \frac{16}{5}$.

Equality when $a = b = c = d = \frac{6}{5}$. Check:

$$4 \cdot \frac{6}{5} + \frac{16}{5} = 8, \quad 4 \cdot \frac{36}{25} + \frac{256}{25} = 16$$

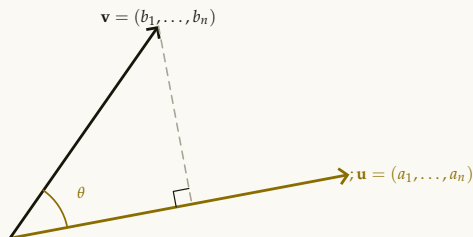
ANSWER $\frac{16}{5}$

5.4 CAUCHY AND SCHWARZ**THEOREM · CAUCHY AND SCHWARZ**

For real numbers,

$$\begin{aligned} (a_1 b_1 + \cdots + a_n b_n)^2 \\ \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2), \end{aligned}$$

with equality exactly when the two lists are proportional.



$$\sum a_i b_i = |u||v| \cos \theta \leq |u||v|, \text{ equal}$$

when v is a non-negative multiple of u

(i) **Titu's form:** for $y_i > 0$,

$$\frac{x_1^2}{y_1} + \cdots + \frac{x_n^2}{y_n} \geq \frac{(x_1 + \cdots + x_n)^2}{y_1 + \cdots + y_n}.$$

(ii) It bounds a linear expression on a sphere or circle: on $x^2 + y^2 = 1$, $(3x + 4y)^2 \leq (9 + 16)(x^2 + y^2) = 25$, so $3x + 4y \leq 5$.

EXAMPLE 5.5

Real numbers x, y, z satisfy $x + 2y + 3z = 14$. Find the least value of $x^2 + y^2 + z^2$.

For $x = y = z$, $x = \frac{7}{3}$ and $x^2 + y^2 + z^2 = \frac{49}{3}$.

By Cauchy and Schwarz,

$$(x + 2y + 3z)^2 \leq (1 + 4 + 9)(x^2 + y^2 + z^2)$$

At least 14. L.H.S. = 196; on dividing by 14, $x^2 + y^2 + z^2 \geq 14$.

Equality when $(x, y, z) = (1, 2, 3)$. Equality needs $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$, i.e., $(x, y, z) = (t, 2t, 3t)$. Then $t + 4t + 9t = 14$ gives $t = 1$.

ANSWER 14

EXAMPLE 5.6

Positive reals a, b, c satisfy $a + b + c = 1$. Find the least value of $\frac{1}{a} + \frac{4}{b} + \frac{9}{c}$.

For $a = b = c = \frac{1}{3}$, the value is $3 + 12 + 27 = 42$.

Writing the numerators as $1^2, 2^2, 3^2$, by Titu's form,

$$\frac{1^2}{a} + \frac{2^2}{b} + \frac{3^2}{c} \geq \frac{(1 + 2 + 3)^2}{a + b + c}$$

At least 36. R.H.S. = $\frac{36}{1} = 36$.

Equality when $a = \frac{1}{6}, b = \frac{1}{3}, c = \frac{1}{2}$. Equality needs $a : b : c = 1 : 2 : 3$ and $a + b + c = 1$. Check: $6 + 12 + 18 = 36$.

ANSWER 36

5.5 MAXIMA AND MINIMA IN INTEGERS

(i) For an integer x : $x^2 \geq |x|$, and $x^2 \geq 4$ unless $x \in \{-1, 0, 1\}$.

(ii) Parity: a sum of an odd number of odd integers is odd.

(iii) A positive integer $k \geq 5$ can be split as $2 + (k - 2)$ with $2(k - 2) > k$. So a product of positive integers with a fixed sum is largest when the parts are 3s, with at most two 2s.

EXAMPLE 5.7

$x_1, x_2, \dots, x_{49}$ are non-zero integers with $x_1 + x_2 + \dots + x_{49} = 0$. Find the least possible value of $x_1^2 + x_2^2 + \dots + x_{49}^2$. (IOQM 2026, Q3)

For 3 numbers: three ± 1 s never add to 0 (odd sum), but $1 + 1 - 2 = 0$, with squares adding to 6.

For 49 numbers: each $x_i^2 \geq 1$. If every x_i is ± 1 , the sum of 49 odd numbers is odd, not 0.

At least 52. So some $x_i^2 \geq 4$, and the other 48 squares are ≥ 1 :

$$\sum x_i^2 \geq 4 + 48 \cdot 1 = 52$$

Example with 52. One 2, twenty-three 1s and twenty-five -1 s:

$$2 + 23 - 25 = 0, \quad 4 + 23 + 25 = 52$$

ANSWER 52

EXAMPLE 5.8

Find the largest product of positive integers whose sum is 20.

Four 5s give the product 625; five 4s give 1024.

In a split with the largest product:

No part is 1: parts 1 and m can be joined into $m + 1 > 1 \cdot m$.

No part is 5 or more: by Section 5.5, such a part k splits as $2 + (k - 2)$ with $2(k - 2) > k$. A part 4 can be replaced by $2 + 2$ (same product).

At most two 2s: three 2s can be replaced by two 3s, as $8 < 9$.

So the parts are 3s with at most two 2s. As $3 \nmid 20$ and $3 \nmid 16$, there is exactly one 2, and $20 - 2 = 3 \cdot 6$.

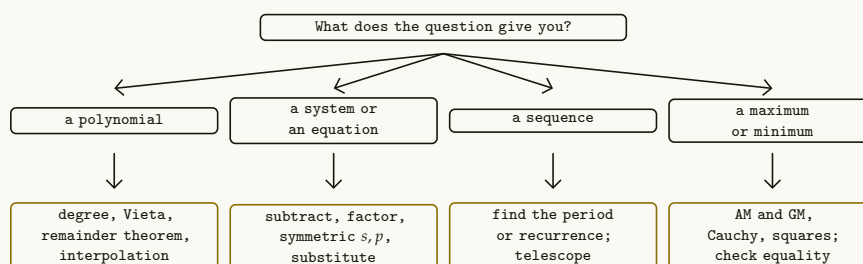
ANSWER $3^6 \cdot 2 = 1458$

WHERE MARKS GO · A BOUND IS NOT YET AN ANSWER

For a least or largest value, give the values at which it is reached. If the equality case of your inequality cannot happen, the bound is not the answer.

5.6 CHOOSING THE METHOD

A question in the paper does not say which chapter it comes from. The chart and the table below match what the question gives with the method to try first.



THE QUESTION GIVES	METHOD TO TRY FIRST	SECTION
an expression to evaluate	factorise; use $a + b + c = 0$, or $x + y$ and xy	1.1, 1.3
values of a polynomial	remainder theorem, or a helper polynomial	2.1, 2.7
roots of a polynomial	Vieta, then reduce powers with the equation	2.2, 2.3
an equation in one unknown	substitute, then square and check	3.3, 3.4
a system	subtract and factorise, or add in pairs	3.6
a functional equation	special values, then a second equation	3.7
a sequence	first terms, period or telescoping	4.3, 4.4
a largest or least value	squares, AM and GM, Cauchy, with equality	5.1 to 5.5

5.7 IOQM 2026 ALGEBRA QUESTIONS

Four of the thirty questions of IOQM 2026 were algebra. Each needed a complete argument: a bound with an example, a degree count, a period, and a full case split.

- (i) **Q3**, least $\sum x_i^2$ when $\sum x_i = 0$: parity, then a bound. Example 5.7.
- (ii) **Q11**, all P with $P(x^2) = P(P(x))$: degrees, then coefficients. Example 2.7.
- (iii) **Q16**, a sum of squares of a sequence: period 4. Example 4.4.
- (iv) **Q21**, a cyclic system: subtract, factorise, then cases. Example 3.9.

5.8 COMMON MISTAKES

Students often lose marks on these points.

- (i) Squaring both sides can add roots. Check every candidate in the original equation.
- (ii) AM and GM need non-negative terms.
- (iii) A least value needs an equality case that actually happens, not only a bound.
- (iv) Dividing by $x - y$ loses the case $x = y$; cancelling a term needs it non-zero.
- (v) A case split must cover every case.
- (vi) $\lfloor -x \rfloor \neq -\lfloor x \rfloor$ unless x is an integer, and $\sqrt{x^2} = |x|$, not x .
- (vii) Vieta counts roots with multiplicity, complex roots included.
- (viii) A function found from special values must be checked in the equation.
- (ix) "Find all" needs every solution and a reason why there are no others.

5.9 A MIXED PROBLEM SET

The exercises below are not arranged by chapter, as in the actual paper. Before solving, decide from the table in 5.6 which method the question needs. Answers with one-line hints are at the end of this part.

EXERCISES 5

- 5.1 Find the least value of $x^2 + \frac{16}{x}$ for $x > 0$.
- 5.2 Positive reals x, y satisfy $x + y = 3$. Find the largest value of xy^2 .
- 5.3 Find the least value of $(a + b)\left(\frac{1}{a} + \frac{4}{b}\right)$ for positive a, b .
- 5.4 Real numbers x, y satisfy $x^2 + y^2 = 25$. Find the largest value of $3x + 4y$.
- 5.5 Find the least value of $x^2 + y^2 + z^2$ for real numbers with $2x - y + 2z = 9$.
- 5.6 Positive reals a, b, c satisfy $abc = 1$. Find the least value of $(1 + a)(1 + b)(1 + c)$.
- 5.7 How many ordered triples (a, b, c) with $a, b, c \in \{1, 2, \dots, 17\}$ satisfy $a^3 + b^3 + c^3 + 2abc = a^2b + a^2c + b^2c + b^2a + c^2a + c^2b$? (*Math Prize for Girls 2012*)
- 5.8 Non-zero reals a, b, c satisfy $a + b + c = 0$. Find $\frac{(a^2 + b^2 + c^2)^2}{a^4 + b^4 + c^4}$.
- 5.9 α, β, γ are the roots of $x^3 - 3x^2 + 1 = 0$. Find $(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)$.
- 5.10 $a_1 = 1$ and $a_{n+1} = a_n + 2n$. Find a_{20} .
- 5.11 Find $\lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \dots + \lfloor \sqrt{100} \rfloor$.
- 5.12 $f(x) + 2f\left(\frac{1}{x}\right) = 3x$ for all real $x \neq 0$. Find $f(2)$.

- 5.13 How many pairs of integers (x, y) satisfy $x^2 + y^2 \leq 2x + 4y - 1$?

PART IV ANSWERS TO THE EXERCISES

Every answer below was checked by computer, by direct search or exact calculation. The hint gives only the first step of the solution; if it is enough, complete the question yourself before reading further.

CHAPTER 1 · IDENTITIES AND MANIPULATION

- 1.1 12 $a + b + c = 0$ gives $a^3 + b^3 + c^3 = 3abc$.
 1.2 80 $s^3 - 3sp = 125 - 45$.
 1.3 123 $t_n = 3t_{n-1} - t_{n-2}$: 2, 3, 7, 18, 47, 123.
 1.4 461 $n(n+1)(n+2)(n+3) + 1 = (n^2 + 3n + 1)^2$ with $n = 20$.
 1.5 6 $11 \pm 6\sqrt{2} = (3 \pm \sqrt{2})^2$.
 1.6 88 pair the factors: $(22 + 2\sqrt{143})(2\sqrt{143} - 22) = 572 - 484$.
 1.7 4030057 with $a = 2007$ the fraction is $a^2 + a + 1$, as $1 + a^4 + (a + 1)^4 = 2(a^2 + a + 1)^2$.
 1.8 11221 $a^4 + b^4 + (a + b)^4 = 2(a^2 + ab + b^2)^2$, $a = 11$, $b = 100$.
 1.9 3 $(x - 2)^2 + (y - 1)^2 = 0$.
 1.10 $-\frac{2}{3}$ three times the equation is $6(y - x)^2 + 2(3x + 1)^2 = 0$, so $x = y = -\frac{1}{3}$.
 1.11 11 $n^{200} < 5^{300} \iff n^2 < 125$.
 1.12 109 $1 + 4 \cdot 16^4 = 545 \cdot 481 = 5 \cdot 109 \cdot 13 \cdot 37$.
 1.13 4 not all equal, so $x + y + z = 0$; with $x = y$, $z = -2x$ and $3x + 3z = 6$.

CHAPTER 2 · POLYNOMIALS

- 2.1 $3x + 1$ $R = rx + s$ with $r + s = 4$ and $-2r + s = -5$.
 2.2 7 $(x^2 + px + 49)(x^2 - px - 41)$; the x term gives $p = -1$, and $k = 8 - p^2$.
 2.3 44 roots $4 - d, 4, 4 + d$; $4(16 - d^2) = 48$; roots 2, 4, 6.
 2.4 $\frac{7}{3}$ $\sum \frac{1}{a} = \frac{e_2}{e_3} = \frac{7/2}{3/2}$.
 2.5 5 $\alpha^3 = \alpha + 1$ gives $p_k = p_{k-2} + p_{k-3}$: $p_{1..5} = 0, 2, 3, 2, 5$.
 2.6 37 $\sum (a - b)^2 = 3 \cdot 3083 - 95^2 = 224$; gaps 8, 4; 25, 33, 37.
 2.7 46 $(4x^2 - 4x - 7)(x^2 - 2x - 2)$; $p = 1 + \sqrt{3}$, $q =$

$$\frac{1+2\sqrt{2}}{2}.$$

- 2.8 $\frac{1}{2}$ $(2x - 1)(x^2 - 2x - 4)$; the roots $1 \pm \sqrt{5}$ are irrational.
 2.9 0 $P(k) - 5 = (k - a)(k - b)(k - c)(k - d)R(k) = 3$ needs four distinct divisors of 3 with product dividing 3.
 2.10 4 0, 1, x , x^2 .
 2.11 39 $P(x) = 3x + (x - 1)(x - 2)(x - 3)(x - 4)$.
 2.12 0 at $x = \omega$: $\omega^{2026} + \omega^{1013} + 1 = \omega + \omega^2 + 1$.
 2.13 1366 $(1 + \omega)^{12} = (1 + \omega^2)^{12} = 1$: $\frac{4096+2}{3}$.

CHAPTER 3 · EQUATIONS AND FUNCTIONAL EQUATIONS

- 3.1 33 $a + c = 2b$, so $N = 13b$, $b = 11$, $a + b + c = 3b$.
 3.2 85 $r + g = 2b$, $w = 13b$, so $b = 3$, $w = 39$; then $r = 1$, $g = 5$, $k = 4$, $N = 42$.
 3.3 63 $\frac{2n+1}{n^2}$ is in lowest terms; $(n + 1)^2 = 1024$, $n = 31$.
 3.4 0 $s = 3$, $p = \frac{5}{2}$, but real a, b need $s^2 \geq 4p$.
 3.5 4 $(s, p) = (5, 0)$ or $(1, -12)$, giving $(5, 0)$, $(0, 5)$, $(4, -3)$, $(-3, 4)$.
 3.6 $x = 0$ $a = 2^x$, $b = 3^x$: $(a - b)^2 + (a - 1)^2 + (b - 1)^2 = 0$.
 3.7 $\frac{3+\sqrt{13}}{2}$ with $u = x - 1$, each term $u + 1 + \frac{3}{u} - 2\sqrt{u+3} = \frac{1}{u}(u - \sqrt{u+3})^2 \geq 0$; so $x = y = z$.
 3.8 6 $|x - 2| + |x - 7| \geq 5$, with equality for $2 \leq x \leq 7$.
 3.9 $\frac{7}{2}$ $[2x] = 2n$ or $2n + 1$; $3n + 1 = 10$ gives $n = 3$, $\{x\} \geq \frac{1}{2}$.
 3.10 $(1, 1, 1)$ $x + y + z = xy + yz + zx = 3$, so $x^2 + y^2 + z^2 = xy + yz + zx$.
 3.11 1 $a + b + c = 0$, so $b^2 - a^2 = a$ reads $(a - b)c = a$; multiply the three.
 3.12 10 the three equations give $f(x) = \frac{6x-2}{7x}$.
 3.13 $-\frac{1}{3}$ with $1 - x$ for x : $f(x) = \frac{2(1-x)^2 - x^2}{3}$.

CHAPTER 4 · SEQUENCES, SERIES AND RECURSIONS

- 4.1 550 $d = 3$, $a_1 = -1$, $S_{20} = 10(-2 + 57)$.
 4.2 502 $(n - m)(n + m + 1) = 4024 = 8 \cdot 503$, factors of opposite parity.
 4.3 $\frac{3}{4}$ $S - \frac{S}{3} = \sum \frac{1}{3^k} = \frac{1}{2}$.

4.4 9 $\left(\frac{n(n+1)}{2}\right)^2 = 45^2$.

4.5 $\frac{101}{200} \prod \frac{k-1}{k} \cdot \prod \frac{k+1}{k} = \frac{1}{100} \cdot \frac{101}{2}$.

4.6 56 $b_n = \frac{1}{a_n a_{n+1}}$ has $b_n = b_{n-1} + n$, $b_0 = 1$.

4.7 1 both equal $1990 - 1991\left(\frac{1}{996} + \dots + \frac{1}{1990}\right)$.

4.8 45 period 6, and $2023 \equiv 1, 2024 \equiv 2 \pmod{6}$.

4.9 2 $2, -1, \frac{1}{2}$ repeat, and $2026 \equiv 1 \pmod{3}$.

4.10 913 the first ten terms add to $11f_7$.

4.11 171 add with weights $1, -2, 4, \dots, 256$: $1 - 2 + 4 - \dots + 256$.

4.12 4895 $F_{10}F_{11} = 55 \cdot 89$.

4.13 58025 roots 2, 3: $a_n = 3^n - 2^n$.

CHAPTER 5 · INEQUALITIES AND PROBLEM SOLVING

5.1 12 $x^2 + \frac{8}{x} + \frac{8}{x} \geq 3\sqrt[3]{64}$, equality at $x = 2$.

5.2 4 AM and GM on $x, \frac{y}{2}, \frac{y}{2}$; $x = 1, y = 2$.

5.3 9 Cauchy: $(a+b)\left(\frac{1}{a} + \frac{4}{b}\right) \geq (1+2)^2$, at $b = 2a$.

5.4 25 $(3x+4y)^2 \leq 25(x^2+y^2)$; at $(3,4)$.

5.5 9 $(2x-y+2z)^2 \leq 9(x^2+y^2+z^2)$; at $(2, -1, 2)$.

5.6 8 $1+a \geq 2\sqrt{a}$ for each factor; $a = b = c = 1$.

5.7 408 $(a+b-c)(b+c-a)(c+a-b) = 0$; 136 triples with $c = a+b$, times 3.

5.8 2 $a+b+c = 0$ gives $a^4+b^4+c^4 = \frac{1}{2}(a^2+b^2+c^2)^2$.

5.9 1 $\alpha + \beta = 3 - \gamma$: the product is $(3-\alpha)(3-\beta)(3-\gamma)$.

5.10 381 $a_{20} = 1 + 2(1 + 2 + \dots + 19)$.

5.11 625 $\lfloor \sqrt{k} \rfloor = m$ for $2m+1$ values; $\sum_{m=1}^9 m(2m+1) + 10$.

5.12 -1 $x = 2$ and $x = \frac{1}{2}$: $f(2) + 2f(\frac{1}{2}) = 6$, $f(\frac{1}{2}) + 2f(2) = \frac{3}{2}$.

5.13 13 $(x-1)^2 + (y-2)^2 \leq 4$: lattice points of a disc of radius 2.

[A NOTE BEFORE YOU CLOSE IT]

Four things worth remembering

Every IOQM answer is an integer from 00 to 99, so it is tempting to guess. Do not guess. An algebra answer is complete only when every step can be reversed or checked: a root found after squaring is checked in the equation, a least value comes with the values that reach it, and a case split covers every case.

Decide the method before you start. The table in section 5.6 sums up this part in eight rows: an expression, values of a polynomial, its roots, an equation, a system, a functional equation, a sequence, a largest or least value. If you cannot say which row a question belongs to, read it again before calculating.

Revise this part three times. The first pass, a few weeks out, is slow and complete, with every exercise attempted. The second, a week out, is only the gold cards and the worked examples. The third, on the morning of the paper, is only the dark cards and the common mistakes of section 5.8, since these are the mistakes most likely to happen in the paper.

Finally, tell us when we are wrong. If a line here does not match what you work out, write to ritesh@iitianforum.com or academics@iitianforum.com with the page and the line. Corrections go into the next edition, and the student who finds one is credited in it.

IOQM HANDBOOK RELEASE

REF IITF-IOQM-2027-HANDBOOK-PART-IV

STATUS PART IV · EDITION 1

Ritesh Raj*IIT Delhi*

Founder, IITian Forum

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